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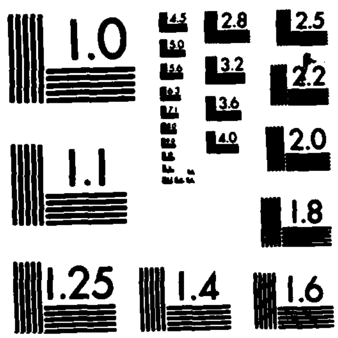

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CONSTRUCTION OF NON-LINEAR MODEL FOR BINARY METAL-MATRIX COMPOSITES

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The research is conducted in response to the need for microstructural-based theories which furnish increased simulation capabilities of metal-matrix composite structures in both static and dynamic regimes with a minimum of model parameters to be experimentally determined. The objective of the research is to develop a nonlinear model of binary metal-matrix composities that will provide greater accuracy than existing models in linear and nonlinear regimes for static and dynamic loading. Furthermore, in an effort to accommodate practical configurations a plate theory is developed for fiber-reinforced composite plates. Progress made includes 1) development and validation of a linear mixture model which accounts for effective moduli and harmonic wave dispersion, 2) testing of methodologies for including nonlinear material responses, and 3) a significant improvement of exisiting laminated composite theories to account for inelastic responses driven by in-plane strains.

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1. SUMMARY

This is the first Annual Report under ONR Contract N00014-84-0468, work unit number NR064-725 conducted by Drs. H. Murakami and G. A. Hegemier at the University of California, San Diego, during the period from July 1, 1984 to June 30, 1985. The cognizant ONR program manager is Dr. A. S. Kushner.

The research is conducted in response to the need for microstructural-based theories which furnish increased simulation capabilities of metal-matrix composite structures in both static and dynamic regimes with a minimum of model parameters to be experimentally determined. The objective of the research is to develop a nonlinear model of binary metal-matrix composites that will provide greater accuracy than existing models in linear and nonlinear regimes for static and dynamic loading. Furthermore, in an effort to accommodate practical structural configurations a plate theory is developed for fiber-reinforced composite plates.

The progress made during the first year's effort toward achieving the above objectives includes: 1) the development and validation of a linear mixture model which accounts for effective moduli and harmonic wave dispersion, 2) testing of methodologies for including nonlinear material responses, and 3) a significant improvement of existing laminated composite theories to account for inelastic responses driven by in-plane strains.

2. RESEARCH OBJECTIVE

The ultimate objective of the research described here is to construct an advanced, nonlinear nonphenomenological model of binary metal-matrix composites that will provide greater accuracy than existing models in linear and nonlinear response regimes for static and dynamic loading. The term 'nonphenomenological' implies a model that is capable of synthesizing the global properties of composites from a knowledge of the matrix and fiber properties, the fiber-matrix interface properties, and the geometry of the fiber reinforcement. In addition to the development of a continuum model, a plate theory is developed for laminated composites in

which each layer consists of a unidirectionally fiber-reinforced metal-matrix composite laminae. This effort is to improve the simulation capabilities of metal-matrix composite structures by introducing the advanced constitutive model to laminated composite plates.

The specific research objectives of the work performed during the present reporting period were as follows:

1. Construction of a linear mixture model - Develop a dispersive linear mixture model for arbitrary wave motion, and perform a validation study of the model.

2. Development of a nonlinear mixture model - Explore an efficient methodology for including inelastic responses of the composite, such as plastic deformation of the matrix, debonding and slip at the fiber-matrix interface, and fiber breakage.

3. Improvement of existing laminated composite plate theories - Develop an improved laminated composite plate theory which can accommodate the above inelastic constitutive model for metal-matrix composite laminae.

3. CURRENT STATUS OF RESEARCH

The progress made during the report period toward achieving the research objectives described earlier is summarized in this section. First, the basic technical approach being followed to meet the objectives is outlined. Then, the progress made toward developing and validating an advanced mixture theory with microstructure for metal-matrix composites is described. Next, the effort to develop a laminated composite plate theory with improved in-plane responses for use in connection with the above constitutive model is summarized.

3.1 Approach

The nonlinear response of metal-matrix composites is largely dominated by complex interaction between the fiber and the metal-matrix. Consequently, an accurate model of metal-matrix composites must be capable of accounting for such interactions. Further, in an effort to

minimize the number and types of tests necessary to define the parameters of a given model, it is highly desirable that it be nonphenomenological, i.e., that the global properties of the composite be synthesized from the constitutive properties of the fiber and matrix, the fiber-matrix interface conditions, and the geometry of fiber reinforcement. A candidate modeling approach that satisfies the above objective is the "mixture theory with microstructure". According to the mixture concept, the fiber and matrix are modeled at each instant of time as superposed continua in space. Each continuum is allowed to undergo individual deformations. The microstructure of an actual composite is then simulated by specifying the nature of the interaction between the continua. The key element for the development of a mixture model for composites with periodic microstructure is an asymptotic procedure called "multivariable asymptotic expansions". This mathematical technique with a "smoothing" operation, leads to the desired mixture forms.

An improved laminated composite plate theory that can simulate in-plane responses accurately is developed by introducing a new displacement microstructure over the thickness of plates and by using Reissner's new mixed variational principle (Reissner, 1984) which automatically yields the shear correction factors of shear deformable plate theories (Reissner, 1946, Mindlin, 1951).

3.2 Development of Linear and Nonlinear Mixture Models for Metal-Matrix Composites

For fibrous composites, wave dispersion has been amply demonstrated via ultrasonic techniques by such investigators as Tauchert and Guzelse (1972), and Sutherland and Lingle (1972).

Simulation of response phenomena associated with the material microstructure, such as wave dispersion, requires a higher-order continuum description. Several such models have been proposed, some phenomenological, some nonphenomenological.

A higher-order continuum model which simulates wave dispersion was first proposed by Achenback and Herrmann (1968) for unidirectionally fiber-reinforced composites. This theory, called the "effective stiffness theory", has been further studied and applied to fibrous composites

by Bartholomew and Torvick (1972), Hlavacek (1975), Achenback (1976), and Aboudi (1981). The aforementioned work concerned linear materials. By modifying the original methodology, Aboudi (1982, 1985) extended the linear model to account for inelastic responses of the composite constituents.

In addition to the effective stiffness modeling concept, a mixture approach has been followed by a number of investigators. A phenomenological version of this model type was adopted by Martin, Bedford and Stern (1971). Deterministic, nonphenomenological mixture theories were introduced by Hegemier, Gurtman and Nayfeh (1973), Hegemier and Gurtman (1974), and Murakami, Maewal and Hegemier (1979). Although capable of simulating nonlinear component responses and interfacial slip, this work was limited to waveguide-type problems. This limitation was removed in the mixture theory developed for laminated composites by Hegemier, Murakami and Maewal (1979), and Murakami, Maewal and Hegemier (1982). In their papers, it was demonstrated that the mixture-type model was capable of simulating harmonic wave dispersion in laminated composites more accurately than the effective stiffness theories. Further, the mixture-type model requires fewer governing equations. The accuracy and efficiency of the mixture theory is due to the use of appropriate displacement and stress microstructural fields, and a judicious smoothing technique. These are obtained by an asymptotic procedure with multiple scales. This procedure yields a series of microboundary value problems (MBVP's) defined over a unit cell, which in term represents the (periodic) microstructure of a composite. The lowest order version of the MBVP method is equivalent to the "O(1) homogenization theory" summarized by Bensoussan, Lions, and Papanicolaou (1978), and Sanchez-Palencia (1980). The latter, while it generates appropriate static moduli, is nondispersive. Simulation of wave dispersion requires at least a theory which is classified as an $O(\epsilon)$ homogenization theory in which ϵ denotes the representative ratio of micro-to-microdimensions of a composite.

To date an $O(\epsilon)$ mixture theory has not been constructed for fibrous composites subject to arbitrary wave motion. Construction and validation of such a 3D model for unidirectional

binary composites with periodic microstructure is the objective of the research. To facilitate this task, the asymptotic procedure with multiple scales noted previously is combined with a variational technique (Murakami, 1985). Following development of the basic equations, the dispersion of time-harmonic waves is studied and the results are compared with experimental data for boron/epoxy (Tauchert and Guzelse, 1972) and tungusten/aluminum (Sutherland and Lingle, 1972) composites. The good correlation obtained with experimental data indicates that the proposed mixture model furnishes a basic tool by which dynamic responses of elastic composites can be investigated.

Following the development of a linear mixture model for metal-matrix composites, an extension of the model to include material nonlinearities has been attempted. Future work will include completion of development of a nonlinear mixture model with validation studies. As part of this effort, the constraint hardening and fiber breakage will be incorporated.

3.3 Development of An Improved Laminated Composite Plate Theory

The application of metal-matrix composites in the form of laminated plates has created a demand for the development of a laminated composite plate theory in which each layer may experience plastic deformation with constraint hardening (Dvorak and coworkers, 1976, 1984), and transverse cracking. In order to simulate the inelastic response of each layer, plate theories should be capable of predicting accurately in-plane strains which yield inelastic responses.

In a series of papers, Pagano (1970a,b) derived exact elasticity solutions for bidirectional composites for the problems of cylindrical bending and simply supported rectangular plates. Pagano showed the importance of the transverse shear effect for the predictions of accurate plate deflections and the necessity of improving assumptions for in-plane displacements, which are assumed to be linear across the thickness of the plate in the Kirchhoff as well as the Reissner-Mindlin shear deformable plate theories. Since the development of laminated plate theories, including the effect of the transverse shear by Yang, Norris and Stavsky (1966) and Whitney and Pagano (1970), many higher order laminated plate theories have been proposed.

Historical accounts of such efforts may be found in the articles by Seide (1980), Bert (1984), and Reddy (1984). However, only a few attempts have been made to improve the in-plane strain responses.

New high-order plate theories have been developed with the help of a new variational principle (Reissner, 1984). The improvement of the in-plane responses is achieved by including a zigzag shaped C^0 function to approximate the thickness variation of the in-plane displacements.

In any approximate plate theory, kinematic assumptions require corresponding constitutive assumptions for transverse stresses. As an example, both the Kirchhoff and the Reissner-Mindlin plate theories adopt a displacement field which satisfies a state of plane strain in the thickness direction. In the Kirchhoff theory the stress field is assumed to be in a state of plane stress which means that all transverse stresses are zero. In the Reissner-Mindlin plate theory only the transverse normal stress is set to be zero in the stress-strain relations. This, in turn, implies that Hook's law, as it is, cannot be used. As a result, a more complicated displacement assumption over the thickness of the plate must be introduced with suitable stress assumptions which are not trivial. Reissner's new mixed variational principle (1984) provides a solution to the above shortcoming; it is a variational principle for arbitrary displacement and transverse stresses, in which the original three dimensional stress-strain relations can be used.

By using the two key elements: a zigzag shaped C^0 interpolation function for the displacement variation over the thickness of the plate and the application of Reissner's new mixed variational principle, new shear deformable laminated plate theories with improved in-plane responses were developed (Murakami, 1985, Toledano and Murakami, 1985). Copies of these references are included in Appendix B, together with the paper which compared the difference of the two plate theories of different order (Murakami and Toledano, 1985).

The overall thrust and implication of the above work is that by the new laminated composite plate theories it is now possible to carry out nonlinear plate analyses in which some layers experience plastic deformations. For the composite plates made of metal-matrix composite laminae the new plate theories can accommodate the new constitutive model developed by the non-phenomenological mixture theory.

4. PUBLICATIONS

The following papers were prepared, and submitted for publication, during the reporting period covered by this report:

1. Murakami, H. and G. A. Hegemier, "A Mixture Model for Unidirectionally Fiber-Reinforced Composites," submitted for publication.

2. Murakami, H., "Laminated Composite Plate Theory with Improved In-Plane Responses," Proceedings of the 1985 PVP Conference, ASME, PVP Vol. 98-2, 1985, pp. 257-263, also submitted to ASME Journal of Applied Mechanics.

3. Toledano, A. and H. Murakami, "A High-Order Laminated Plate Theory with Improved In-Plane Responses," submitted for publication.

   The following paper was prepared for a cancelled symposium during the reporting period and will be submitted to an appropriate symposium.

4. Murakami, H. and Toledano, A., "An Improved Laminated Composite Plate Theory," will be submitted.

5. INTERACTIONS

The following is a list of the presentations at meetings and conferences by the principal investigators which occurred during the reporting period on issues related to the research done under the present contract:

1. Murakami, H., "A Mixture Model for Metal-Matrix Composites," oral presentation at the Tenth Annual Mechanics of Composite Review, Dayton, Ohio, October 15-17, 1984.

2. Murakami, H., "Some Basic Inelastic Response Features of the New Endochronic Theory," oral presentation at the 21st Annual Meeting of the Society of Engineering Science, VPI, Blacksburg, Virginia, October 15, 1984.

3. Murakami, H., "Laminated Composite Plate Theory with Improved In-Plane Responses," oral presentation at the 1985 PVP conference, ASME, New Orleans, Louisiana, June 24-26, 1985.

6. LIST OF PROFESSIONAL PERSONNEL

Scientific personnel supported by the contract during the reporting period are

1. Principal Investigators: Dr. H. Murakami, Assistant Professor of Applied Mechanics, and Dr. G. A. Hegemier, Professor of Applied Mechanics.

2. Research Assistants: Mr. Akira Akiyama, Mr. Albert Toledano, and Mr. Thomas Impelso, PhD students in Engineering Sciences (Applied Mechanics).

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APPENDIX A

A MIXTURE MODEL
FOR UNIDIRECTIONALLY FIBER-REINFORCED COMPOSITES

by

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ABSTRACT

A binary mixture theory with microstructure is constructed for unidirectionally fiber-reinforced elastic composites. Model construction is based on an asymptotic scheme with multiple scales and the application of Reissner's new mixed variational principle (1984). In order to assess the accuracy of the model, comparison of the mixture model predictions with available experimental data on dispersion of harmonic waves is made for boron/epoxy and tungusten/aluminum composites. Formulas for the effective moduli are also presented, and the results are compared with test data and other available predictions.

1. Introduction

With the advent of high strength and stiffness fibers such as boron and carbon, and the development of techniques for binding such materials to plastic or metal, fibrous composites have become important elements of modern structures. Such composites, due to their microstructural heterogeneity, may exhibit response phenomena for some environments that are not observed for homogeneous materials. An example of these phenomena for dynamic environments is wave dispersion, and understanding of which is important both from the standpoints of direct response prediction and indirect analyses associated with such topics as nondestructive testing. For fibrous composites, wave dispersion has been amply demonstrated via ultrasonic techniques by such investigators as Tauchert and Guzelse (1972), and Sutherland and Lingle (1972).

Simulation of response phenomena associated with the material microstructure, such as wave dispersion, requires a higher-order continuum description. Several such models have been proposed, some phenomenological, some nonphenomenological.

A higher-order continuum model which simulates wave dispersion was first proposed by Achenback and Herrmann (1968) for unidirectionally fiber-reinforced composites. This theory, called the "effective stiffness theory", has been further studied and applied to fibrous composites by Bartholomew and Torvick (1972), Hlavacek (1975), Achenback (1976), and Aboudi (1981). The aforementioned work concerned linear materials. By modifying the original methodology, Aboudi (1982, 1983) extended the linear model to account for inelastic responses of the composite constituents.

In addition to the effective stiffness modeling concept, a mixture approach has been followed by a number of investigators. A phenomenological version of this model type was adopted by Martin, Bedford and Stern (1971). Deterministic, nonphenomenological mixture theories were introduced by Hegemier, Gurtman and Nayfeh (1973), Hegemier and Gurtman (1974), ar. Murakami, Maewal and Hegemier (1979). Although capable of simulating nonlinear component responses and interfacial slip, this work was limited to waveguide-type problems. This limitation was removed in the mixture theory developed for laminated composites by Hegemier, Murakami and Maewal (1979), and Murakami, Maewal and Hegemier (1982). In their papers, it was demonstrated that the mixture-type model was

capable of simulating harmonic wave dispersion in laminated composites more accurately than the effective stiffness theories. Further, the mixture-type model requires fewer governing equations. The accuracy and efficiency of the mixture theory is due to the use of appropriate displacement and stress microstructural fields, and a judicious smoothing technique. These are obtained by an asymptotic procedure with multiple scales. This procedure yields a series of microboundary value problems (MBVP's) defined over a unit cell, which in turn represents the (periodic) microstructure of a composite. The lowest order version of the MBVP method is equivalent to the "O(1) homogenization theory" summarized by Bensoussan, Lions, and Papanicolaou (1978), and Sanchez-Palencia (1980). The latter, while it generates appropriate static moduli, is nondispersive. Simulation of wave dispersion requires at least a theory which is classified as an $O(\epsilon)$ homogenization theory in which ϵ denotes the representative ratio of micro-to-macrodimensions of a composite.

To date an $O(\epsilon)$ mixture theory has not been constructed for fibrous composites subject to arbitrary wave motion. Construction and validation of such a 3D model for unidirectional binary composites with periodic microstructure are the objective of this paper. To facilitate this task, the asymptotic procedure with multiple scales noted previously is combined with a variational technique (Murakami, 1985). Following development of the basic equations, the dispersion of time-harmonic waves is studied and the results are compared with experimental data for boron/epoxy (Tauchert and Guzelse, 1972) and tungusten/aluminum (Sutherland and Lingle, 1972) composites. The good correlation obtained with experimental data indicates that the proposed mixture model furnishes a basic tool by which dynamic responses of elastic composites can be investigated. While the model construction procedure is applicable to inelastic component response and interface slip, extension and investigation of the nonlinear problem is deferred to later publications.

2. Formulation

Consider a domain $\overline{V}$ which contains a uniaxial periodic array of fibers embedded in the matrix, as shown in Fig. 1. Let a rectangular reference system $\overline{x}_1, \overline{x}_2, \overline{x}_3$ be selected with $\overline{x}_1$ in the axial direction of the fibers. In the $\overline{x}_2, \overline{x}_3$-plane, a typical cell that represents the geometrical microstructure of the

composite is shown in Fig. 2 for a hexogonal array.

For notational convenience forms $(\ )^{(\alpha)}$, $\alpha = 1,2$ denote quantities associated with material α with $\alpha = 1$ representing fiber and $\alpha = 2$ matrix. Cartesian indicial notation will be employed in which Latin indices range from 1 to 3 and repeated indices imply the summation convention unless otherwise stated. In addition, the notations $(\bar{\ })_{,j} \equiv \partial(\bar{\ })/\partial \bar{x}_j$ and $(\bar{\ })_{,t} \equiv \partial(\bar{\ })/\partial \bar{t}$ will be employed in which $\bar{t}$ represents time. Quantities of the form $(\bar{\ })$ and $(\ )$ denote dimensional and nondimensional variables, respectively.

The governing relations for the displacement vector $\bar{u}_i^{(\alpha)}$ and the stress tensor $\bar{\sigma}_{ij}^{(\alpha)}$ in the two constituents are:

(a) Equations of motion

$$\bar{\sigma}_{ji,j}^{(\alpha)} = \bar{\rho}^{(\alpha)} \bar{u}_{i,tt}^{(\alpha)} \quad , \quad \bar{\sigma}_{ji}^{(\alpha)} = \bar{\sigma}_{ij}^{(\alpha)} \tag{1}$$

where $\bar{\rho}^{(\alpha)}$ is the mass density;

(b) Constitutive relations

$$\bar{\sigma}_{ij}^{(\alpha)} = \bar{\lambda}^{(\alpha)} \delta_{ij}\, e_{kk}^{(\alpha)} + 2\bar{\mu}^{(\alpha)}\, e_{ij}^{(\alpha)} \tag{2}$$

where $\bar{\lambda}^{(\alpha)}$, $\bar{\mu}^{(\alpha)}$ are Lame's constants, $e_{ij}^{(\alpha)}$ is the infinitesimal Cauchy strain, and δ_{ij} is the Kronecker delta;

(c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2}\, (\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}) \quad ; \tag{3}$$

(d) Interface continuity relations

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)} \quad , \quad \bar{\sigma}_{ji}^{(1)} \nu_j^{(1)} = \bar{\sigma}_{ji}^{(2)} \nu_j^{(1)} \qquad \text{on} \quad \bar{\mathscr{I}} \tag{4}$$

where $\nu_1^{(1)} \equiv 0$ on the fiber-matrix interface $\bar{\mathscr{I}}$;

(e) Initial conditions at $\bar{t} = 0$ and appropriate boundary data along the boundary $\partial \bar{V}$.

Conditions (a) - (e) define a well posed initial boundary value problem. However, due to the large number of fiber-matrix interfaces the direct solution to this problem is extremely difficult. The

objective of the subsequent analysis is to alleviate such difficulties by deriving a set of partial differential equations with constant coefficients whose solution can be utilized to approximate the solution of the problem. To this end, it will be convenient to nondimensionalize the basic equations by using the following quantities:

$\overline{\Lambda}$	typical macrosignal wavelength
$\overline{\Delta}$	typical fiber spacing or cell dimension
$\overline{C}_{(m)}$, $\overline{\rho}_{(m)}$	reference wave velocity and macrodensity
$\overline{E}_{(m)} \equiv \overline{\rho}_{(m)} \overline{C}^2_{(m)}$	reference modulus
$\overline{t}_{(m)} \equiv \overline{\Lambda}/\overline{C}_{(m)}$	typical macrosignal travel time
$\epsilon \equiv \overline{\Delta}/\overline{\Lambda}$	ratio of micro-to-macrodimensions.

With the aid of the above notation, nondimensional variables are now introduced according to

$$(x_1 , x_2 , x_3) = (\overline{x}_1 , \overline{x}_2 , \overline{x}_3)/\overline{\Lambda} \quad , \quad t = \overline{t}/\overline{t}_{(m)} \quad ,$$

$$(\lambda , \mu)^{(\alpha)} = (\overline{\lambda} , \overline{\mu})^{(\alpha)}/\overline{E}_{(m)} \quad , \quad \rho^{(\alpha)} = \overline{\rho}^{(\alpha)}/\overline{\rho}_{(m)} \quad . \tag{5}$$

With the variables defined according to (5), the material properties are seen to be periodic in the x_2, x_3-plane in which the periodicity of the fiber lattice structure may be defined by the cell. It is expected that stress and deformation fields will vary significantly with respect to two basic length scales: (1) a "global" or "macro" length typical of the body size or loading condition, and (2) a "micro" length typical of "cell" planar dimensions. Further, it is expected that these scales will differ by at least one order of magnitude in most cases. This suggests the use of multivariable asymptotic techniques (Bensoussan, Lion and Papanicolaou, 1978, Hegemier, Murakami and Maewal, 1979, Sanchez-Palencia, 1980). This approach commences by introducing new independent microvariables according to

$$x_i^* \equiv x_i/\epsilon \quad . \tag{6}$$

Therefore, all field variables are considered to be functions of the microvariables x_2^* and x_3^*, as well as the macrovariables x_i , $i = 1-3$:

$$f(x_1, x_2, x_3, t) = f^*(x_1, x_2, x_3, x_2^*, x_3^*, t \, ; \epsilon) \tag{7a}$$

Spacial derivatives of a function f then takes the form

$$\frac{\partial}{\partial x_i} f\,(x_k\,,t) = \frac{\partial}{\partial x_i} f^*\,(x_k\,,x_j^*\,,t\,;\epsilon)$$

$$+ \frac{1}{\epsilon}\frac{\partial}{\partial x_i^*} f^*\,(x_k\,,x_j^*\,,t\,;\epsilon) \tag{7b}$$

where $\partial(\ )/\partial x_1^* \equiv 0$. By introducing the notation $(\ )_{,j^*} \equiv \partial(\ )/\partial x_i^*$ equation (7b) can be rewritten as:

$$f_{,i} = f^*_{,i} + \frac{1}{\epsilon} f^*_{,j^*} \quad . \tag{7c}$$

In the sequel f^* will be written as f for notational simplicity.

The operations (7), when applied to all field variables, lead to the following "synthesized" governing field relations:

(a) Equations of motion

$$\sigma_{ji,j}^{(\alpha)} + \frac{1}{\epsilon}\,\sigma_{ji,j^*}^{(\alpha)} = \rho^{(\alpha)}\,u_{i,tt}^{(\alpha)} \quad , \ \sigma_{ji}^{(\alpha)} = \sigma_{ij}^{(\alpha)} \quad ; \tag{8}$$

(b) Constitutive relations

$$\sigma_{ij}^{(\alpha)} = \lambda^{(\alpha)}\,\delta_{ij}\,e_{kk}^{(\alpha)} + 2\mu^{(\alpha)}\,e_{ij}^{(\alpha)} \quad ; \tag{9}$$

(c) Strain-displacement relations

$$e_{ij}^{(\alpha)} = \frac{1}{2}\left\{u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)} + \frac{1}{\epsilon}\,(u_{i,j^*}^{(\alpha)} + u_{j,i^*}^{(\alpha)})\right\} \quad ; \tag{10}$$

(d) Interface continuity conditions

$$u_i^{(1)} = u_i^{(2)} \quad , \ \sigma_{ji}^{(1)}\nu_j^{(1)} = \sigma_{ji}^{(2)}\nu_j^{(1)} \quad \text{on } \mathcal{I} \,. \tag{11}$$

At this point, the variation of field variables which satisfy the periodicity with respect to x_i^* is assumed. According to this condition field variables take equal values on opposite sides of the cell boundary. The premise allows one to analyze a single cell in an effort to determine the distribution of any field variable with respect to the microcoordinates x_i^*. The x^*-periodicity condition is motivated by the Floquet and

Block theorems (Brillouin, 1946) for harmonic wave in periodic structures. Certainly, it eliminates boundary layer effects. However, it is expected to provide a good model for the global wave phenomena in fibrous composites with periodic microstructure.

For the construction of a mixture model it is convenient to cast the field equations in a variational form by using the Reissner new mixed variational principle (Reissner, 1984). In the Reissner variational principle the variations of displacement, strain with (10) as definition and transverse stresses, i.e., all stress-components except $\sigma_{11}^{(\alpha)}$, are considered. Thus, it is convenient to rewrite the constitutive relation (9) in terms of the axial strain $e_{11}^{(\alpha)}$ and the transverse stresses:

$$\sigma_{11}^{(\alpha)} \equiv (\lambda + 2\mu)^{(\alpha)} e_{11}^{(\alpha)} + \lambda^{(\alpha)} \left\{ e_{22}^{(\alpha)}(\cdots) + e_{33}^{(\alpha)}(\cdots) \right\} ,$$

$$\begin{bmatrix} e_{22}^{(\alpha)}(\cdots) \\ e_{33}^{(\alpha)}(\cdots) \end{bmatrix} \equiv \begin{bmatrix} (\lambda + 2\mu)^{(\alpha)} & \lambda^{(\alpha)} \\ \lambda^{(\alpha)} & (\lambda + 2\mu)^{(\alpha)} \end{bmatrix}^{-1} \left(\begin{bmatrix} \sigma_{22} \\ \sigma_{33} \end{bmatrix}^{(\alpha)} - \lambda^{(\alpha)} e_{11}^{(\alpha)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) ,$$

$$[2e_{23}^{(\alpha)}(\cdots) \ , \ 2e_{31}^{(\alpha)}(\cdots) \ , \ 2e_{12}^{(\alpha)}(\cdots)]$$

$$\equiv \frac{1}{\mu^{(\alpha)}} [\sigma_{23}^{(\alpha)} \ , \ \sigma_{31}^{(\alpha)} \ , \ \sigma_{12}^{(\alpha)}] \tag{12}$$

Using the equations of motion (8), Gauss' theorem, and the $x^\bullet$-periodicity condition, it can be demonstrated that the Reissner mixed variational principle, applied to the synthesized fields by the multivariable representation, takes the form:

$$\iiint_V \left[\sum_{\alpha^{(\alpha)}} \iint_{A^{(\alpha)}} \left\{ \delta e_{11}^{(\alpha)} \sigma_{11}^{(\alpha)} + \delta e_{22}^{(\alpha)} \hat{\sigma}_{22}^{(\alpha)} + \delta e_{33}^{(\alpha)} \hat{\sigma}_{33}^{(\alpha)} + 2\delta e_{23}^{(\alpha)} \hat{\sigma}_{23}^{(\alpha)} \right. \right.$$

$$+ 2\delta e_{31}^{(\alpha)} \hat{\sigma}_{31}^{(\alpha)} + 2\delta e_{12}^{(\alpha)} \hat{\sigma}_{12}^{(\alpha)}$$

$$+ \delta\hat{\sigma}_{22}^{(\alpha)} \left(u_{2,2}^{(\alpha)} + \frac{1}{\epsilon} u_{2,2^\bullet}^{(\alpha)} - e_{22}^{(\alpha)}(\cdots) \right) + \delta\hat{\sigma}_{33}^{(\alpha)} \left(u_{3,3}^{(\alpha)} + \frac{1}{\epsilon} u_{3,3^\bullet}^{(\alpha)} - e_{33}^{(\alpha)}(\cdots) \right)$$

$$+ \delta\hat{\sigma}_{23}^{(\alpha)} \left(u_{2,3}^{(\alpha)} + u_{3,2}^{(\alpha)} + \frac{1}{\epsilon} u_{2,3}^{(\alpha)} + \frac{1}{\epsilon} u_{3,2^\bullet}^{(\alpha)} - 2e_{23}^{(\alpha)}(\cdots) \right)$$

$$+ \delta\hat{\sigma}_{31}^{(\alpha)} \left(u_{1,3}^{(\alpha)} + u_{3,1}^{(\alpha)} + \frac{1}{\epsilon} u_{1,3^\bullet}^{(\alpha)} - 2e_{31}^{(\alpha)}(\cdots) \right)$$

$$+ \delta \hat{\sigma}_{12}^{(\alpha)} \Big(u_{1,2}^{(\alpha)} + u_{2,1}^{(\alpha)} + \frac{1}{\epsilon} u_{1,2^*}^{(\alpha)} - 2 e_{12}^{(\alpha)} (\cdots) \Big) \Big\} dx_2^* \, dx_3^*$$

$$+ \int_{\mathscr{I}} \Big\{ (\delta u_i^{(2)} - \delta u_i^{(1)}) \overset{\nu}{T}{}_i^* + \delta \overset{\nu}{T}{}_i^* (u_i^{(2)} - u_i^{(1)}) \Big\} ds^* \Big] \, dx_1 \, dx_2 \, dx_3$$

$$= \iiint_V \Big\{ \sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \delta u_i^{(\alpha)} (- \rho^{(\alpha)} u_{i,tt}^{(\alpha)}) dx_2^* \, dx_3^* \Big\} \, dx_1 \, dx_2 \, dx_3$$

$$+ \iint_{\partial V_T} \Big(\sum_{\alpha=1}^{2} \iint_{A^{(\alpha)}} \delta u_i^{(\alpha)} \, \overset{\nu}{T}{}_i^{(\alpha)} \, dx_2^* \, dx_3^* \Big) dA \quad , \tag{13}$$

where $A^{(\alpha)}$ denotes the x_2^*, x_3^*-domain of the cell occupied by material α (Fig. 2), $\hat{\sigma}_{ij}^{(\alpha)}$ is used for the approximate transverse stresses, $\overset{\nu}{T}{}_i^{(\alpha)}$ denotes the traction vector on the surface ∂V_T where the traction is specified, ds^* is an infinitesimal line element on $\mathscr{I}$, and dA is an infinitesimal surface element on the boundary of $V : \partial V$. In (13) basic variables are the displacement $u_i^{(\alpha)}$, the transverse stresses $\hat{\sigma}_{ij}^{(\alpha)}$ and the *interface* traction vector $\overset{\nu}{T}{}_i^*$. The Euler-Lagrange equations of (13) include (8a), (11a), (12), and

$$\overset{\nu}{T}{}_i^* \equiv \hat{\sigma}_{ji}^{(\alpha)} \nu_j^{(1)} \qquad \text{on} \quad \mathscr{I} \; . \tag{14}$$

The above variational equation (13) furnishes a tool with which a mixture model can be obtained with appropriate trial displacement and transverse stress fields. The basic requirement for the variables is the x^*-periodicity condition on the cell boundary ∂A. The microstructural variation of the trial functions can be obtained by the asymptotic procedure (Murakami, Maewal and Hegemier, 1981).

3. Asymptotic Analysis

The premise that the composite macrodimension is much larger than the microdimension, $\epsilon << 1$, and the form of scaled equations (8) and (10), suggest the expansion of the dependent variables in the asymptotic series:

$$\{ u_i , \sigma_{ij} \}^{(\alpha)} (x_k , x_l^* , t ; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \{ u_{i(n)} , \sigma_{ij(n)} \}^{(\alpha)} (x_k , x_l^* , t) \quad . \tag{15}$$

If (15) is substituted into (8)-(11) and the coefficients of different powers of ϵ are equated to zero, a sequence of problems defined on the cell is obtained. The first of the equations in this sequence furnishes

$$u^{(\alpha)}_{i(o),j^*} = 0 \quad , \quad \sigma^{(\alpha)}_{ji(o),j^*} = 0 \quad . \tag{16}$$

Equation (16a) implies that $u^{(\alpha)}_{i(o)}$ is independent of x^*_j and yields with the zero-th order expansion of (11a):

$$u^{(\alpha)}_{i(o)} = U_{i(o)}\ (x_k\ , t) \quad . \tag{17}$$

The remaining systems of equations obtained from (8)-(10) are, for $n \geqslant 0$:

$$\sigma^{(\alpha)}_{ji(n+1),j^*} = \rho^{(\alpha)} u^{(\alpha)}_{i(n),tt} - \sigma^{(\alpha)}_{ji(n),j}\ , \sigma^{(\alpha)}_{ji(n)} = \sigma^{(\alpha)}_{ij(n)} \quad , \tag{18}$$

$$\sigma^{(\alpha)}_{ij(n)} = \lambda^{(\alpha)}\delta_{ij}\ e^{(\alpha)}_{kk(n)} + 2\mu^{(\alpha)} e^{(\alpha)}_{ij(n)} \quad , \tag{19}$$

$$e^{(\alpha)}_{ij(n)} = \frac{1}{2}\ (u^{(\alpha)}_{i(n),j} + u^{(\alpha)}_{j(n),i} + u^{(\alpha)}_{i(n+1),j^*} + u^{(\alpha)}_{j(n+1),i^*}) \quad . \tag{20}$$

To be added to the foregoing are the interface conditions and the x^*-periodicity conditions for $n \geqslant 0$:

$$u^{(1)}_{i(n)} = u^{(2)}_{i(n)} \quad , \ \sigma^{(1)}_{ji(n)}\nu^{(1)}_j = \sigma^{(2)}_{ji(n)}\nu^{(1)}_j \quad \text{on} \quad \mathcal{I} \ , \tag{21}$$

$$u^{(2)}_{i(n)} \ \text{ and } \ \sigma^{(2)}_{ji(n)}\nu^{(2)}_j \ \text{ are } \ x^*\text{—periodic on } \partial A \quad . \tag{22}$$

The first set of microboundary value problems (MBVP's) for $\sigma^{(\alpha)}_{ji(o)}$ and $u^{(\alpha)}_{i(1)}$, called the O(1) MBVP's, is defined by (16b), (18b), (19)-(20), (21b), (22b) with $n = 0$, and (21a), (22a) with $n = 1$. The O(1) MBVP's are excited by $U_{i(o),j}$. Similarly, a sequency of MBVP's is defined for each n from (18)-(22). With appropriate integrability and normalization conditions, higher order terms may be computed by solving the MBVP's. In particular, the O(1) MBVP's are the ones solved for the O(1) homogenization theory proposed by Bensoussan, Lion and Papanicolaou (1978) and Sanchez-Palencia (1980), and, also, form the basis of the mixture theory which may be classified as an O(ϵ) homogenization theory. The asymptotic approach yields the microstructures of displacement and stress fields after solving a multitude of MBVP's which are complicated.

In order to use the approximate solutions of the MBVP's in the course of developing a mixture model, and to ease the burden of solving the MBVP's exactly, a variational procedure was adopted by Murakami (1985) for laminated composites with the help of the Reissner new mixed variational principle (Reissner, 1984). A similar approach is adopted here for fibrous composites. To obtain the lowest order mixture theory by using (13), it is necessary to obtain trial displacement and transverse stresses to O(ϵ). In the sequel, the trial functions are obtained for a hexagonal cell with a concentric cylinders approximation as shown in Fig. 2. In Fig. 2, (r,θ) are micropolar coordinates:

$$r = \sqrt{x_2^{*2} + x_3^{*2}} \quad , \quad \tan\theta = x_3^*/x_2^* \quad , \tag{23}$$

by which $r = 1$ constitutes the cell boundary and $r = \sqrt{n^{(1)}}$, denotes the interface $\mathcal{I}$. The quantities $n^{(\alpha)}$ indicate the volume fraction of material α and satisfy

$$n^{(1)} + n^{(2)} = 1 \quad . \tag{24}$$

In terms of the polar coordinates the x^*-periodicity conditions for a hexagonal cell with the concentric cylinders approximation reduce to the form:

$$f(x_{,} r, \theta, t) = f(x_k, r, \pi + \theta, t) \quad \text{at} \quad r = 1 \quad . \tag{25}$$

4. Trial Displacements and Transverse Stresses

The O(1) stress and O(ϵ) displacement fields are obtained by solving the O(1) MBVP's which are defined by (16b), (18b), (19)-(22) and (24). These MBVP's are excited by $U_{i(o),j}$. The exact solution of $u_i\{^\alpha_1\}$ is furnished in the Appendix. For the mixture formulation it is convenient to introduce an O(ϵ) displacement variable which represents $U_{i(o),j} + U_{j(o),i}$ according to:

$$\overset{j}{S}_i(x_k, t) \equiv \frac{1}{\epsilon A} \int_{\mathcal{I}} u_i^{(\alpha)} \nu_j^{(1)} ds^* = \frac{1}{A} \int_{\mathcal{I}} u_i\{^\alpha_1\} \nu_j^{(1)} ds^* \tag{26}$$

where $A\,(=\pi)$ is the area of the cell. Due to the fact that $u_i\{^\alpha_1\}$ is excited by $U_{i(o),j} + U_{j(o),i}$ one obtains

$$\overset{3}{S}_2 = \overset{2}{S}_3 \quad . \tag{27}$$

Equation (27) can also be obtained if one substitutes the exact $u_{i\{1\}}^{\{\alpha\}}$ in the Appendix into (26) and eliminates $U_{i(o),j}$. To render the analysis tractable, it is preferable to utilize an approximate form of the exact solution for $u_{i\{1\}}^{\{\alpha\}}$. The exact solution indicates that the following form of the O(ϵ) displacement yields a good approximation:

$$u_{i\{1\}}^{\{\alpha\}}(x_k, x_j^*, t) = \overset{2}{S}_i(x_k, t) g^{(\alpha)}(r) \cos\theta + \overset{3}{S}_i(x_k, t) g^{(\alpha)}(r) \sin\theta \tag{28a}$$

where

$$g^{(1)}(r) = \frac{r}{n^{(1)}} \quad , \quad g^{(2)}(r) = \frac{1}{n^{(2)}}\left(-r + \frac{1}{r}\right) \quad . \tag{28b}$$

Anticipating the O(ϵ^2) difference of the average of $u_i^{(\alpha)}$ on $A^{(\alpha)}$, equations (17) and (27) yield the following trial displacement field:

$$u_i^{(\alpha)}(x_k, x_j^*, t) = U_i^{(\alpha)}(x_k, t) + \epsilon\, u_{i\{1\}}^{\{\alpha\}}(x_k, x_j^*, t) \tag{29}$$

where $u_{i\{1\}}^{\{\alpha\}}$ is defined by (28). Equations (29) and (28) indicate that the mixture displacement variables are $U_i^{(1)}$, $U_i^{(2)}$, $\overset{2}{S}_i$ and $\overset{3}{S}_i$ with the constraint (27).

By using (29) in (19) with $n = 0$ and considering the O(ϵ^2) differences of the average transverse stresses, the O(1) trial stress field may be expressed as:

$$\begin{bmatrix} \hat{\sigma}_{22(o)} \\ \hat{\sigma}_{33(o)} \\ \hat{\sigma}_{23(o)} \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \tau_{22}(x_k, t) \\ \tau_{33}(x_k, t) \\ \tau_{23}(x_k, t) \end{bmatrix}^{(\alpha)} + \frac{\delta_{\alpha 2}}{r^2}\left\{ t_{22}^{\{2\}}(x_k, t)\begin{bmatrix} \cos 2\theta \\ \cos 2\theta \\ 0 \end{bmatrix} + t_{33}^{\{2\}}(x_k, t)\begin{bmatrix} \cos 2\theta \\ -\cos 2\theta \\ \sin 2\theta \end{bmatrix} + t_{23}^{\{2\}}(x_k, t)\begin{bmatrix} \sin 2\theta \\ \sin 2\theta \\ 0 \end{bmatrix} \right\} \tag{30a}$$

$$\begin{bmatrix} \hat{\sigma}_{31(o)} \\ \hat{\sigma}_{12(o)} \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \tau_{31}(x_k, t) \\ \tau_{12}(x_k, t) \end{bmatrix}^{(\alpha)} + \frac{\delta_{\alpha 2}}{r^2}\left\{ t_{12}^{\{2\}}(t_k, t)\begin{bmatrix} \sin 2\theta \\ \cos 2\theta \end{bmatrix} + t_{31}^{\{2\}}(x_k, t)\begin{bmatrix} \cos 2\theta \\ -\sin 2\theta \end{bmatrix} \right\} \tag{30b}$$

In order to define the O(ϵ) trial stress field it is convenient to define the O(ϵ) stress variable according to

$$P_i(x_k, t) \equiv \frac{1}{\epsilon A}\int_{\mathcal{G}} \sigma_{ji}^{(\alpha)} \nu_j^{(1)} ds^* = \frac{1}{A}\int_{\mathcal{G}} \sigma_{ji\{1\}}^{\{\alpha\}} \nu_j^{(1)} ds^* \quad . \tag{31}$$

If one integrates (8a) over $A^{(\alpha)}$ and utilizes the x^*-periodicity condition, one obtains the mixture momentum equations:

$$n^{(\alpha)}\, \sigma_{ji,j}^{(a\alpha)} + (-1)^{\alpha+1}\, P_i = n^{(\alpha)}\, \rho^{(\alpha)}\, u_{i,tt}^{(a\alpha)} \tag{32}$$

where the average operation is defined by

$$f^{(a\alpha)}(x_k, t) \equiv \frac{1}{n^{(\alpha)}A} \iint_{A^{(\alpha)}} f^{(\alpha)}(x_k, x_j^*, t)\; dx_2^* dx_3^* \tag{33}$$

From (32) it can be seen that P_i represents an interaction body force between the two constituents across the interface. Also, the form of (32) with P_i defined by (31) satisfies the integrability condition adopted by the O(1) MBVP's for $\sigma_{ij(1)}^{(\alpha)}$ which are defined by (18)-(20) with appropriate n's and (31). As an $O(\epsilon)$ trial stress field which satisfies (31b) one may use the following approximate fields:

$$\begin{bmatrix} \hat{\sigma}_{22(1)} \\ \hat{\sigma}_{33(1)} \\ \hat{\sigma}_{23(1)} \end{bmatrix}^{(\alpha)} = -\frac{1}{4} \left\{ P_2(x_k, t) g^{(\alpha)}(r) \cos\theta \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + P_3(x_k, t) g^{(\alpha)}(r) \sin\theta \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\} , \tag{34a}$$

$$\begin{bmatrix} \hat{\sigma}_{31(1)} \\ \hat{\sigma}_{12(1)} \end{bmatrix}^{(\alpha)} = -\frac{1}{2}\, P_1(x_k, t) g^{(\alpha)}(r) \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} . \tag{34b}$$

As a result, the trial transverse stresses are expressed as:

$$\hat{\sigma}_{ij}^{(\alpha)} \equiv \hat{\sigma}_{ij(o)}(x_k, x_i^*, t) + \epsilon\, \hat{\sigma}_{ij(1)}^{(\alpha)}(x_k, x_i^*, t) \tag{35}$$

where $\hat{\sigma}_{ij(o)}^{(\alpha)}$ and $\hat{\sigma}_{ij(1)}^{(\alpha)}$ are defined by (30) and (34), respectively.

5. Mixture Equations

By substituting the displacement and transverse stress trial functions defined by (29) and (35), respectively, into the Reissner variational equation (13), one obtains the following relations as the Euler-Lagrange equations:

(a) Equations of motion

$$n^{(\alpha)}\sigma_{ji,j}^{(a\alpha)} + (-1)^{\alpha+1} P_i = n^{(\alpha)}\rho^{(\alpha)}\, U_{i,tt}^{(\alpha)} \quad , \; i = 1-3 \quad , \tag{36}$$

$$\overset{2}{M}_{ji,j} + \frac{1}{\epsilon^2}\left(\sigma_{2i}^{(2a)} - \sigma_{2i}^{(1a)} + R_{2i}^{(2)}\right) = I\, \overset{2}{S}_{i,tt} \quad , \; i = 1,2 \quad , \tag{37a,b}$$

$$\overset{3}{M}_{ji,j} + \frac{1}{\epsilon^2}\,(\sigma_{3i}^{(2a)} - \sigma_{3i}^{(1a)} + R_{3i}^{(2)}) = I\,\overset{3}{S}_{i,tt} \quad , \; i = 1,3 \quad , \tag{37c,d}$$

$$\overset{3}{M}_{22,2} + \overset{2}{M}_{33,3} + \frac{1}{\epsilon^2}\,(\sigma_{23}^{(2a)} - \sigma_{23}^{(1a)} + R_{23}^{(2)}) = I\,\overset{3}{S}_{2,tt} \quad . \tag{37e}$$

where

$$\sigma_{ij}^{(\alpha a)} \equiv \frac{1}{n^{(\alpha)}A}\iint_{A^{(\alpha)}} \sigma_{ij}^{(\alpha)} dx_2^* \, dx_3^* \quad ,$$

$$\epsilon\,(\overset{2}{M}_{ij} \, , \overset{3}{M}_{ij}) \equiv \frac{1}{A}\sum_{\alpha=1}^{2}\iint_{A^{(\alpha)}} \sigma_{ij}^{(\alpha)} g^{(\alpha)}(\cos\theta \, , \sin\theta)\, dx_2^* \, dx_3^* \quad , \tag{38}$$

and

$$I \equiv \sum_{\alpha=1}^{2} h^{(\alpha)}\rho^{(\alpha)} \quad , \; h^{(1)} = \frac{1}{4} \; , \; h^{(2)} = \frac{-1}{4n^{(2)}}\,(2 + n^{(2)} + \frac{2}{n^{(2)}}\ln n^{(1)}) \quad ; \tag{39}$$

(b) Constitutive relations

$$\begin{bmatrix}\sigma_{22}\\ \sigma_{33}\end{bmatrix}^{(\alpha a)} = \begin{bmatrix}\tau_{22}\\ \tau_{33}\end{bmatrix}^{(\alpha)} = \begin{bmatrix}\lambda + 2\mu & \lambda\\ \lambda & \lambda + 2\mu\end{bmatrix}^{(\alpha)} \begin{bmatrix}U_{2,2}^{(\alpha)} + (-1)^{\alpha+1}\,\overset{2}{S}_2/n^{(\alpha)}\\ U_{3,3}^{(\alpha)} + (-1)^{\alpha+1}\,\overset{3}{S}_3/n^{(\alpha)}\end{bmatrix} + \lambda^{(\alpha)}\,U_{1,1}^{(\alpha)}\begin{bmatrix}1\\ 1\end{bmatrix} \quad ,$$

$$\begin{bmatrix}\sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}^{(\alpha a)} = \begin{bmatrix}\tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix}^{(\alpha)} = \mu^{(\alpha)}\begin{bmatrix}U_{2,3} + U_{3,2}\\ U_{3,1} + U_{1,3}\\ U_{1,2} + U_{2,1}\end{bmatrix}^{(\alpha)} + \frac{(-1)^{\alpha+1}}{n^{(\alpha)}}\begin{bmatrix}2\,\overset{3}{S}_2\\ \overset{3}{S}_1\\ \overset{2}{S}_1\end{bmatrix} \quad ; \tag{40}$$

$$P_1 = \beta_1\,[(U_1^{(2)} - U_1^{(1)}/\epsilon^2 + (h/2)\{\overset{2}{S}_{1,2} + \overset{2}{S}_{2,1}) + (\overset{3}{S}_{3,1} + \overset{3}{S}_{1,3})\}]$$

$$P_2 = \beta_2[(U_2^{(2)} - U_2^{(1)})/\epsilon^2 + \gamma\,\overset{2}{S}_{1,1} + h\,(\overset{2}{S}_{2,2} + \overset{3}{S}_{2,3})]$$

$$P_3 = \beta_3[(U_3^{(2)} - U_3^{(1)})/\epsilon^2 + \gamma\,\overset{3}{S}_{1,1} + h\,(\overset{3}{S}_{2,2} + \overset{3}{S}_{3,3})] \tag{41}$$

where

$$\beta_1 = 1/\{\sum_{\alpha=1} h^{(\alpha)}/(2\mu^{(\alpha)})\} \; , \beta_2 = \beta_3 = 1/\{\sum_{\alpha=1} h^{(\alpha)}/(\lambda + \mu)^{(\alpha)}\} \quad ,$$

$$\gamma = \sum_{\alpha=1}^{2} h^{(\alpha)}\lambda^{(\alpha)}/(\lambda + \mu)^{(\alpha)} \quad , \quad h = \sum_{\alpha=1}^{2} h^{(\alpha)} \quad ; \tag{42}$$

$$\overset{2}{M}_{22} = hP_2\ ,\ \overset{2}{M}_{33} = \frac{h}{2}\,P_2\ ,\ \overset{2}{M}_{12} = \frac{h}{2}\,P_1\ ,\ \overset{3}{M}_{22} = \frac{h}{2}\,P_3\quad ,$$

$$\overset{3}{M}_{33} = hP_3\ ,\ \overset{3}{M}_{31} = \frac{h}{2}\,P_1\ ,\ \overset{2}{M}_{23} = \overset{2}{M}_{23} = \overset{2}{M}_{31} = \overset{3}{M}_{23} = \overset{3}{M}_{12} = 0 \tag{43}$$

where it is understood that

$$\overset{2}{M}_{ij} = \overset{2}{M}_{ji}\quad ,\quad \overset{3}{M}_{ij} = \overset{3}{M}_{ji}\quad ; \tag{44}$$

$$R_{21}^{(2)} \equiv t_{12}^{(2)}/n^{(2)}\ ,\ R_{22}^{(2)} \equiv (t_{22}^{(2)}/2 + t_{33}^{(2)})/n^{(2)}\ ,\ R_{31}^{(2)} \equiv -\,t_{31}^{(2)}/n^{(1)}\quad ,$$

$$R_{33}^{(2)} \equiv (-\,t_{22}^{(2)}/2 + t_{33}^{(2)}\,)/n^{(1)} \tag{45}$$

and

$$t_{12}^{(2)} = -\,\mu^{(2)}\,\overset{2}{S}_1/n^{(2)}\ ,\ t_{31}^{(2)} = \mu^{(2)}\,\overset{3}{S}_1/n^{(2)}\quad ,$$

$$t_{22}^{(2)} = -\,(\lambda + \mu)^{(2)}(\overset{2}{S}_2 - \overset{3}{S}_3))/n^{(2)}\ ,\ t_{33}^{(2)} = -\,\mu^{(2)}(\overset{2}{S}_2 + \overset{3}{S}_3)/n^{(2)}\quad ,$$

$$t_{23}^{(2)} = -\,(\lambda + \mu)^{(2)}\,(2\,\overset{3}{S}_2)/n^{(2)}\quad . \tag{46}$$

The remaining constitutive relations associated with $\sigma_{11}^{(\alpha)}$ are obtained from (12a); the results are

$$\sigma_{11}^{(\alpha a)} = (\lambda + 2\mu)^{(\alpha)}\,U_{1,1}^{(\alpha)} + \lambda^{(\alpha)}\left\{U_{2,2}^{(\alpha)} + U_{3,3}^{(\alpha)} + (-1)^{\alpha+1}(\overset{2}{S}_2 + \overset{3}{S}_3)/n^{(\alpha)}\right\}\quad , \tag{47}$$

$$\begin{bmatrix}\overset{2}{M}_{11}\\ \overset{3}{M}_{11}\end{bmatrix} = \sum_{\alpha=1}^{2} h^{(\alpha)}(\lambda + 2\mu)^{(\alpha)} - \frac{\lambda^{(\alpha)2}}{(\lambda + \mu)^{(\alpha)}}\begin{bmatrix}\overset{2}{S}_{1,1}\\ \overset{3}{S}_{1,1}\end{bmatrix} + \gamma\begin{bmatrix}P_2\\ P_3\end{bmatrix}\quad . \tag{48}$$

The associated boundary conditions are on ∂V

$$n^{(\alpha)}\sigma_{ji}^{(\alpha a)}\nu_j = \overset{\nu}{T}_i^{(\alpha p)}\qquad \text{or}\qquad \delta U_i^{(\alpha)} = 0\qquad ,\quad i=1,2,3\ , \tag{49}$$

$$\overset{2}{M}_{ji}\nu_j = \overset{2\nu}{T}_i\qquad \text{or}\qquad \delta\overset{2}{S}_i = 0\qquad ,\quad i=1,2\ , \tag{50a}$$

$$\overset{3}{M}_{ji}\nu_j = \overset{3\nu}{T}_i\qquad \text{or}\qquad \delta\overset{3}{S}_i = 0\qquad ,\quad i=1,3\ . \tag{50b}$$

where

$$\overset{\nu}{T}{}_i^{(\alpha p)} \equiv \frac{1}{A} \iint_{A^{(\alpha)}} \overset{\nu}{T}{}_i^{(\alpha)} \, dx_2^* \, dx_3^* \quad ,$$

$$\epsilon\left(\overset{2\nu}{T}_i \, , \, \overset{3\nu}{T}_i\right) = \frac{1}{A} \sum_{\alpha=1} \overset{\nu}{T}{}_i^{(\alpha)} g^{(\alpha)}(\cos\theta \, , \, \sin\theta) \, dx_2^* \, dx_3^* \quad . \tag{51}$$

Equations (36)-(50) and the initial condition

$$U_i^{(\alpha)} \quad , \quad U_{i,l}^{(\alpha)} \quad , \quad \overset{j}{S}_i \quad , \quad \overset{j}{S}_{i,J} \qquad \text{at} \quad t = 0 \quad \text{on} \quad V \tag{52}$$

define a well posed initial boundary value problem with respect time t and the macrocoordinates x_k.

6. Harmonic Wave Dispersion Spectra

In an attempt to test the accuracy of the mixture model, the phase velocity and group velocity spectra of the mixture theory have been compared with available experimental data for time harmonic waves. For the comparison harmonic waves which are propagating at an arbitrary angle of incidence in a full space of the following form are considered:

$$[U_1^{(1)} \, , U_1^{(2)} \, , U_2^{(1)} \, , U_2^{(2)} \, , U_3^{(1)} \, , U_3^{(2)} \, , \overset{2}{S}_1/ik \, , \overset{2}{S}_2/ik \, , 2\overset{3}{S}_2/ik \, , \overset{3}{S}_3/ik]^T$$

$$= \exp\left\{ ik\,(x_1 \cos\phi + x_2 \sin\phi \cos\theta + x_3 \sin\phi \sin\theta) - i\omega t \right\} \underset{\sim}{U}^* \tag{53}$$

where

$$\underset{\sim}{U}^* = [\dot{U}_1^{(1)} \, , \dot{U}_1^{(2)} \, , \dot{U}_2^{(1)} \, , \dot{U}_2^{(2)} \, , \dot{U}_3^{(1)} \, , \dot{U}_3^{(2)} \, , \overset{2}{s}_1 \, , \overset{2}{s}_2 \, , 2\overset{3}{s}_2 \, , \overset{3}{s}_1 \, , \overset{3}{s}_3]^T \tag{54}$$

and $[\ ]^T$ denotes the transpose of $[\ ]$. In (53) $\dot{U}_i^{(\alpha)}$ and $\overset{j}{s}_i$ are constant amplitudes, k denotes the wave number, ω represents angular frequency, ϕ is the azimuth measured from the x_1 axis, and θ is the longitude; the direction of the wave propagation may be best represented by the wave vector $\underset{\sim}{k}$:

$$\underset{\sim}{k} = k\,[\cos\phi \, , \sin\phi \cos\theta \, , \sin\phi \sin\theta]^T \quad . \tag{55}$$

Substitution of (53) into (36) and (37), which are expressed by the displacement variables with (40)-(46), yields an eigenvalue problem for $\epsilon\omega$ of the form:

$$[K]\dot{\underset{\sim}{U}} = (\epsilon\omega)^2[M]\dot{\underset{\sim}{U}} \tag{56}$$

where $[K]$ and $[M]$ and are 11×11 real symmetric matrices, the elements of which are functions of the mixture constants and the wave vector. Furthermore, $[M]$ is a diagonal matrix. Upon calculation of the eigenvalue $\epsilon\omega$ for a given $\epsilon\underset{\sim}{k}$, one obtains the phase velocity C_p as

$$C_p = (\epsilon\omega)/(\epsilon k) \quad . \tag{57}$$

For each computed eigen pairs $(\epsilon\omega\ ,\ \dot{\underset{\sim}{U}})_k$, $k = 1,2,\cdots 11$ the group velocity

$$C_g = \frac{d\omega}{dk} \tag{58}$$

can be obtained by taking the derivative of (56) with respect to ϵk:

$$[K']_k\ \dot{\underset{\sim}{U}}_k^* = \left\{2(\epsilon\omega)C_g\,[M] + (\epsilon\omega)^2[M']\right\}_k \dot{\underset{\sim}{U}}_k \quad . \tag{59}$$

For the $k\,th$ eigenpair equation (59) yields

$$(C_g)_k = \frac{\dot{\underset{\sim}{U}}^T\{[K'] - (\epsilon\omega)^2[M']\}\dot{\underset{\sim}{U}}}{2(\epsilon\omega)_k\,(\dot{\underset{\sim}{U}}^T[M]\dot{\underset{\sim}{U}})_k} \quad . \tag{60}$$

In the subsequent simulation a typical cell dimension $\bar{\Delta}$ was chosen to be a cell radius by introducing the concentric cylinders approximation of the equal area. The reference elastic modulus and density used for the scaling are

$$\bar{E}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\,\bar{E}^{(\alpha)} \quad , \quad \bar{\rho}_{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\,\bar{\rho}^{(\alpha)} \tag{61}$$

where $\bar{E}^{(\alpha)}$ is Young's modulus. The dimensional frequency $\nu(H_z)$ can be computed from $\epsilon\omega$ by

$$\nu = (\epsilon\omega)\sqrt{\bar{E}_{(m)}/\bar{\rho}_{(m)}}/(2\pi\bar{\Delta}) \quad . \tag{62}$$

Numerical results are presented for a boron-epoxy composite, for which experimental results were presented by Tauchert and Guzelse (1972) for a waveguide case $\phi = 0°$ and a wavefiect case $\phi = 90°$. The material properties are summarized in Table 1 in which the values for Poisson's ratio are estimated. In the simulation $\bar{\Delta}$ was computed from the fiber diameter ($= 2\sqrt{n^{(1)}\bar{\Delta}}$) which was 1.016×10^{-4} m. The group velocity spectra for a waveguide case $\phi = \theta = 0°$ are shown in Fig. 3 for two acoustic modes: a "gross" longitudinal mode and a "gross" shear mode. In the figure the same symbols as the reference of Tauchert and Guzelse are used for the experimental data points. It is noted that reasonable agreement is achieved for the waveguide case in which pronounced dispersion is observed. The group velocity spectra for a wavereflect case $\phi = 90°$, $\theta = 0°$ in which the wave vector is normal to the fiber axis are shown in Fig. 4 with the experimental data. The figure includes three acoustic modes: a "gross" longitudinal wave (P-mode), a "gross" vertically polarized shear wave (SV-mode), and a "gross" horizontally polarized shear wave (SH-mode). The sets of experimental data correspond to the "gross" P-mode and the "gross" SH-mode. It is noted that there are significant deviations from the "gross" SH-mode, but the overall agreement is not unsatisfactory if one admits the scarcity of the experimental data and the difficulties associated with the measurement of shear wave velocities. It was reported by Tauchert and Guzelse (1972) that a shear wave exhibited extremely high damping of the pulse. A similar observation and the scatter of shear wave data were reported by Sachse (1974) who conducted modulus measurements of boron/epoxy composites by using pulse-echo techniques. He concluded that "the measurements of the present investigation indicate that shear waves propagating along and across fibers in the composite materials tested do not always propagate at the same speed."

Sutherland and Lingle (1972) reported phase velocity measurements for tungusten/aluminum composites whose material properties are shown in Table 2. The equivalent cell radius $\bar{\Delta}$ was computed from the given fiber spacings which yield the area of a typical cell $\bar{A}$ ($= \pi \bar{\Delta}^2$) 0.579×10^{-6} m^2. Figure 5 shows the phase velocity vs. frequency relation for the "gross" longitudinal mode. A reasonable agreement is observed between the experimental data and the theoretical prediction.

7. Effective Moduli

The O(1) homogenization theory which yields the effective moduli of composites can be obtained by taking the limit of $\epsilon \rightarrow 0$ and equating the constituents' displacements

$$U_i^{(1)} = U_i^{(1)} = U_i \tag{63}$$

By introducing the above constraints, equations (36) yield

$$\sigma_{ji,j}^{(m)} = \rho^{(m)} U_{i,tt} \tag{64}$$

where

$$\sigma_{ij}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \sigma_{ij}^{(\alpha a)} \quad , \quad \rho^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} \rho^{(\alpha)} \quad . \tag{65}$$

Equations (37) yield

$$\sigma_{2i}^{(2a)} - \sigma_{2i}^{(1a)} + R_{2i}^{(2)} = 0 \quad , \quad i=1,2,3 \tag{66a}$$

$$\sigma_{3i}^{(2a)} - \sigma_{3i}^{(1a)} + R_{3i}^{(2)} = 0 \quad , \quad i=1,3 \quad . \tag{66b}$$

By eliminating $\overset{j}{S}_i$ by (66), equations (65a), (40) and (47) with (63) furnish

$$\underset{\sim}{\sigma}^{(m)} = [E^{(m)}]\underset{\sim}{e}^{(m)} \tag{67}$$

where

$$\underset{\sim}{\sigma}^{(m)} = [\sigma_{11}^{(m)} , \sigma_{22}^{(m)} , \sigma_{33}^{(m)} , \sigma_{23}^{(m)} , \sigma_{31}^{(m)} , \sigma_{12}^{(m)}]^T \quad ,$$

$$\underset{\sim}{e}^{(m)} = [U_{1,1} , U_{2,2} , U_{3,3} , U_{2,3} + U_{3,2} , U_{3,1} + U_{1,3} , U_{1,2} + U_{2,1}]^T \quad , \tag{68}$$

and $[E^{(m)}]$ is the effective modulus matrix with transverse isotropy due to the concentric cylinders approximation and is defined in the Appendix.

The formulas for the effective moduli (B2) are assessed by comparing the results with the experimental data reported by Datta and Ledbetter (1983) for boron/aluminum composites. The results are shown in Table 3 in which the moduli computed from the effective stiffness theories for the square cell by Achenback (1976) and for the hexagonal cell by Hlavacek (1976) are included by using the formulas

reported by Datta and Ledbetter (1983). The comparison has revealed that all high-order theories yield almost similar results. It can be easily shown that the formulas for the effective moduli yield values which fall between the upper and the lower bounds obtained by Hashin and Rosen (1964) for fiber-reinforced composites.

8. Concluding Remarks

An asymptotic mixture theory with multiple scales was applied to unidirectionally fiber-reinforced elastic composites with periodic microstructure. In the model construction, Reissner's new mixed variational principle was applied to the synthesized fields with multivariable field representations. In order to assess the accuracy of the model the mixture dispersion spectra were compared with the experimental data obtained for the boron/epoxy composite by Tauchert and Guzelse (1972) and for the tungusten/aluminum composite by Sutherland and Lingle (1972).

A satisfactory correlation with the experimental data indicates that the proposed mixture model furnishes a basic tool by which dynamic responses of the composite structures can be investigated.

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Appendix

A. Exact $u_{i(1)}^{(\alpha)}$ of the O(1) MBVP's:

$$u_{1(1)}^{(\alpha)} = G\, g^{(\alpha)}(r)\,\{(U_{1(0),2} + U_{2(0),1})\cos\theta + (U_{3(0),1} + U_{1(0),3})\sin\theta\} \quad ,$$

$$u_{2(1)}^{(\alpha)} = b_o\,(U_{2(0),2} + U_{3(0),3} + \hat{\lambda}\, U_{1(0),1})g^{(\alpha)}(r)\cos\theta$$

$$+ a_2^{(\alpha)}\Big[\{g^{(\alpha)}(r)\cos\theta + \hat{\kappa}^{(\alpha)}\, g_{II}^{(\alpha)}(r)\cos 3\theta\}(U_{2(0),2} - U_{3(0),3})$$

$$+ \{g^{(\alpha)}(r)\sin\theta + \hat{\kappa}^{(\alpha)}\, g_{II}^{(\alpha)}(r)\sin 3\theta\}(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ b_2^{(\alpha)}\Big[\{3(1-\hat{\kappa}^{(\alpha)})g_I^{(\alpha)}(r)\cos\theta + (1+\hat{\kappa}^{(\alpha)})g_{III}^{(\alpha)}(r)\cos 3\theta\}(U_{2(0),2} - U_{3(0),3})$$

$$+ \{3(1-\kappa^{(\alpha)})g_I^{(\alpha)}(r)\sin\theta + (1+\hat{\kappa}^{(\alpha)})g_{III}^{(\alpha)}(r)\sin 3\theta\}(U_{2(0),3} + U_{3(0),2})\Big] \quad ,$$

$$u_{3(1)}^{(\alpha)} = b_o\,(U_{2(0),2} + U_{3(0),3} + \hat{\lambda}\, U_{1(0),1})g^{(\alpha)}(r)\sin\theta$$

$$+ a_2^{(\alpha)}\Big[\{g^{(\alpha)}(r)\sin\theta - \hat{\kappa}^{(\alpha)} g_{II}^{(\alpha)}\sin 3\theta\}(-U_{2(0),2} + U_{3(0),3})$$

$$+ \{g^{(\alpha)}(r)\cos\theta - \hat{\kappa}^{(\alpha)} g_{II}^{(\alpha)}(r)\cos 3\theta\}(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ \{g^{(\alpha)}(r)\cos\theta - \hat{\kappa}^{(\alpha)} g_{II}^{(\alpha)}(r)\cos 3\theta\}(U_{2(0),3} + U_{3(0),2})\Big]$$

$$+ b_2^{(\alpha)}\Big[\{3(1-\kappa^{(\alpha)})g_I^{(\alpha)}(r)\sin\theta - (1+\kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\sin 3\theta\}(-U_{2(0),2} + U_{3(0),3})$$

$$+ \{3(1-\kappa^{(\alpha)})g_I^{(\alpha)}(r)\cos\theta - (1+\kappa^{(\alpha)})g_{III}^{(\alpha)}(r)\cos 3\theta\}(U_{2(0),3} + U_{3(0),2})\Big] \qquad \text{(A1)}$$

where

$$b_o \equiv \{(\lambda+\mu)^{(2)} - (\lambda+\mu)^{(1)}\}/(2d_1) \ , \ d_1 \equiv \sum_{\alpha=1}^{2} (\lambda+\mu)^{(\alpha)}/n^{(\alpha)} + \mu^{(2)}/(n^{(1)}n^{(2)})$$

$$G \equiv -(\mu^{(1)} - \mu^{(2)})/d_3 \ , \ d_3 \equiv \sum_{\alpha=1}^{2} \mu^{(\alpha)}/n^{(\alpha)} + \mu^{(2)}/(n^{(1)}n^{(2)}) \quad ,$$

$$\hat{\lambda} \equiv (\lambda^{(1)} - \lambda^{(2)})/\{(\lambda+\mu)^{(1)} - (\lambda+\mu)^{(2)}\} \ .$$

$$\kappa^{(\alpha)} \equiv (\lambda+2\mu)^{(\alpha)}/\mu^{(\alpha)} \ , \ \hat{\kappa}^{(\alpha)} \equiv (1-\kappa^{(\alpha)})/(1+\kappa^{(\alpha)}) \qquad \text{(A2)}$$

$$g_I^{(1)}(r) = g_{III}^{(1)}(r) = r^3 \quad , \quad g_{II}^{(1)}(r) = 0$$

$$g_I^{(2)}(r) = -r^3 + r^{-1} \quad , \quad g_{II}^{(2)}(r) = (-r^{-1} + r^{-3})/n^{(2)} \quad ,$$

$$g_{III}^{(2)}(r) = r^3 - (1+\kappa^{(2)})^{-2}\{4(1-\kappa^{(2)}+\kappa^{(2)2})r^{-3} - 3(1-\kappa^{(2)})^2 r^{-1}\} \quad , \tag{A3}$$

In Eqs. (A1) $a_2^{(\alpha)}$ and $b_2^{(\alpha)}$ are obtained by solving the linear equations for $\underset{\sim}{x} = [a_2^{(1)}, b_2^{(1)}, a_2^{(2)}, b_2^{(2)}]^T$:

$$\underset{4\times 4}{[A]}\ \underset{4\times 1}{\underset{\sim}{x}} = \underset{4\times 1}{\underset{\sim}{B}} \tag{A4}$$

where

$$A_{11} = 1\,,\ A_{21} = 0\,,\ A_{31} = \mu^{(1)}/n^{(1)}\,,\ A_{41} = 0$$

$$A_{12} = 3(1-\kappa^{(1)})n^{(1)2} \quad , \quad A_{22} = (1+\kappa^{(1)})n^{(1)2}$$

$$A_{32} = -A_{42} = 3n^{(1)}\mu^{(1)}(1-\kappa^{(1)}) \quad , \quad A_{13} = -1 \quad , \quad A_{23} = -\hat{\kappa}^{(\alpha)}/n^{(\alpha)} \quad ,$$

$$A_{33} = \mu^{(2)}/n^{(2)}\{1 - \hat{\kappa}^{(\alpha)}/n^{(1)}\} \quad , \quad A_{43} = 3\hat{\kappa}^{(2)}\mu^{(2)}/n^{(1)2}$$

$$A_{14} = -3n^{(2)}(1 + n^{(1)})(1 - \kappa^{(2)})$$

$$A_{24} = (1 + \kappa^{(2)})n^{(1)2} - (4 - 3n^{(1)})(1 - \kappa^{(2)} + \kappa^{(2)2})/\{n^{(1)}(1 + \kappa^{(2)})\}$$

$$A_{34} = 3\mu^{(2)}(1 - \kappa^{(2)})(n^{(1)} - \hat{\kappa}^{(2)}/n^{(1)})$$

$$A_{44} = -3\mu^{(2)}(1 - \kappa^{(2)})(n^{(1)} + 3\hat{\kappa}^{(2)}/n^{(1)}) + 12\mu^{(2)}(1 - \kappa^{(2)} + \kappa^{(2)2})/\{n^{(1)2}(1 + \kappa^{(2)})\} \quad , \tag{A5}$$

and

$$B_1 = B_2 = B_4 = 0 \quad , \quad B_3 = -(\mu^{(1)} - \mu^{(2)})/2 \quad . \tag{A6}$$

It is interesting to note that for most of practical composites $b_2^{(\alpha)}$, $\alpha = 1,2$ are small compared with $a_2^{(\alpha)}$.

B. The Definition of $[E^{(m)}]$ in (67)

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}^{(m)} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0\\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0\\ E_{12} & E_{23} & E_{22} & 0 & 0 & 0\\ 0 & 0 & 0 & E_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & E_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & E_{55}\end{bmatrix}^{(m)} \begin{bmatrix} U_{1,1}\\ U_{2,2}\\ U_{3,3}\\ U_{2,3}+U_{3,2}\\ U_{3,1}+U_{1,3}\\ U_{1,2}+U_{2,1}\end{bmatrix} \tag{B1}$$

where

$$E_{11}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}(\lambda+2\mu)^{(\alpha)} - (\lambda^{(1)}-\lambda^{(2)})^2/d_1 \quad ,$$

$$E_{12}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\lambda^{(\alpha)} - (\lambda^{(1)}-\lambda^{(2)})\{(\lambda+\mu)^{(1)}-(\lambda+\mu)^{(2)}\}/d_1 \quad ,$$

$$E_{22}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}(\lambda+2\mu)^{(\alpha)} - \{(\lambda+\mu)^{(1)}-(\lambda+\mu)^{(2)}\}^2/d_1 - (\mu^{(1)}-\mu^{(2)})^2/d_2 \quad ,$$

$$E_{23}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\lambda^{(\alpha)} - \{(\lambda+\mu)^{(1)}-(\lambda+\mu)^{(2)}\}^2/d_1 + (\mu^{(1)}-\mu^{(2)})^2/d_2 \quad ,$$

$$E_{44}^{(m)} = (E_{22}^{(m)} - E_{23}^{(m)})/2 \quad ,$$

$$E_{55}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)}\mu^{(\alpha)} - (\mu^{(1)}-\mu^{(2)})^2/d_3 \quad , \tag{B2}$$

and where

$$d_2 = \sum_{\alpha=1}^{2} \mu^{(\alpha)}/n^{(\alpha)} + (\lambda+\mu)^{(2)}/(2n^{(1)}n^{(2)}) \quad . \tag{B3}$$

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Table 1. Material Properties of the Boron/Epoxy Composite Tested by Tauchert and Guzelse (1972)

	Volume Fraction $n^{(\alpha)}$	Young's Modulus $\bar{E}^{(\alpha)}$	Poisson's Ratio $\nu^{(\alpha)}$	Mass Density $\bar{\rho}^{(\alpha)}$
$^{(1)}$Boron	0.54	379.2 GPa (55×10^6 psi)	0.18	2682 kg/m^3 (251×10^{-6} lb sec^2/in^4)
$^{(2)}$Epoxy	0.46	5.033 GPa (0.73×10^6 psi)	0.40	1261 kg/m^3 (118×10^{-6} lb sec^2/in^4)

Table 2. Material Properties of the Tungsten/Aluminum Composite Tested by Sutherland and Lingle (1972)

	Volume Fraction $n^{(\alpha)}$	Young's Modulus $\bar{E}^{(\alpha)}$	Poisson's Ratio $\nu^{(\alpha)}$	Mass Density $\bar{\rho}^{(\alpha)}$
$^{(1)}$Tungsten	0.022	398 GPa	0.28	19194 kg/m^3
$^{(2)}$Aluminum	0.978	71.0 GPa	0.34	2700 kg/m^3

Table 3. Comparison of Effective Moduli of a Boron/Aluminum Unidirectionally Fiber-Reinforced Composite in Units of 10^{11} N/m^2

	Data[a]	Mixture Model	Square Cell[a] Model	Hexagonal Cell[a] Model
$E_{11}^{(m)}$	2.450	2.551	2.480	2.551
$E_{22}^{(m)}$	1.825	1.868	1.856	1.872
$E_{23}^{(m)}$	0.779	0.661	-----	0.661
$E_{12}^{(m)}$	0.604	0.578	-----	0.578
$E_{44}^{(m)}$	0.526	0.604	-----	0.606
$E_{55}^{(m)}$	0.566	0.559	0.451	0.561

[a] After Datta and Ledbetter (1983)

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Fig. 2 A typical cell

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Fig. 4 Group velociry spectra of wavereflect modes for a boron/epoxy composite (Tauchert and Guzelse,1972)

Fig. 5 A phase velocity spectrum of a longitudinal wave-reflect mode for a tungusten/aluminum composite (Sutherland and Lingle,1972)

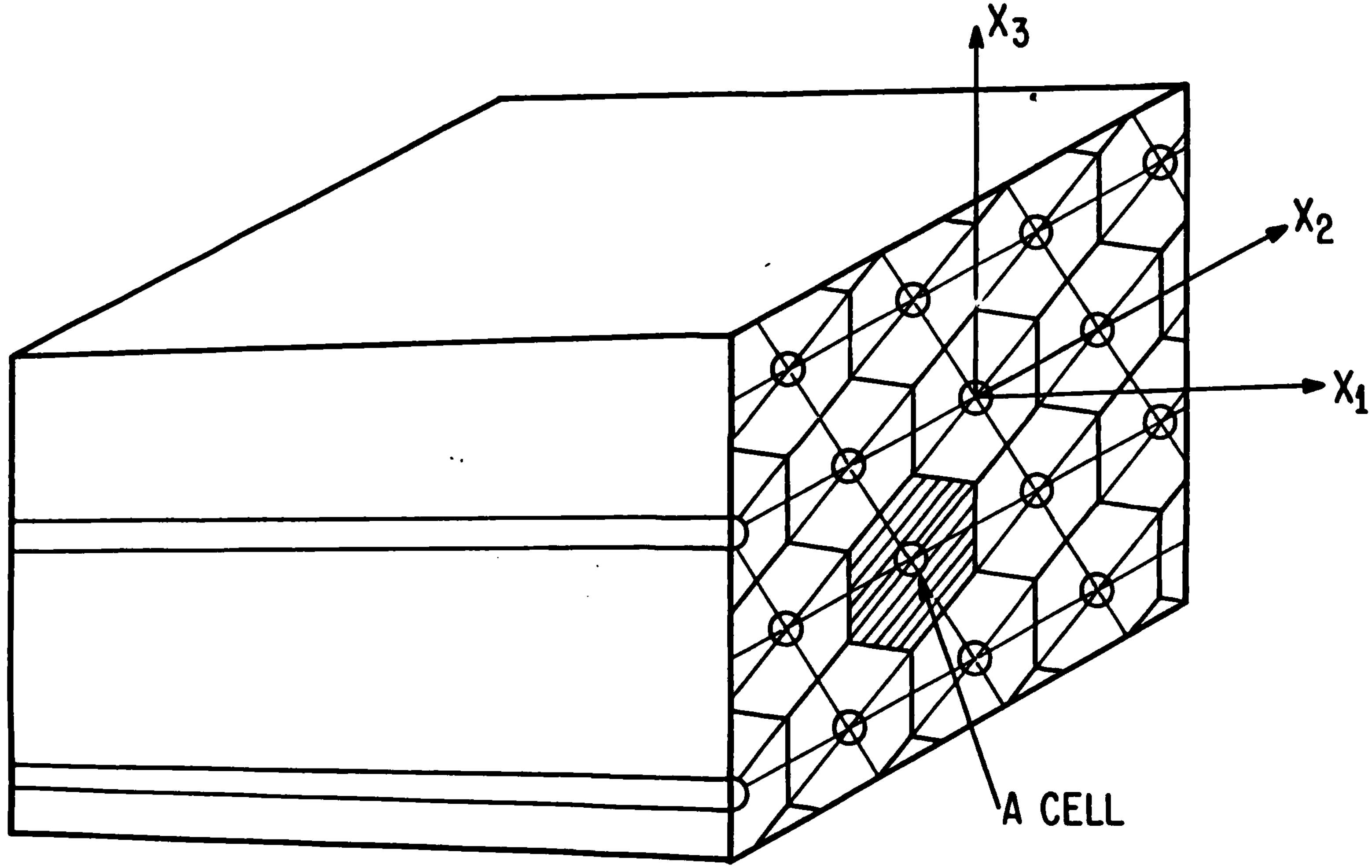

Figure 1. A unidirectionally fiber-reinforced composite

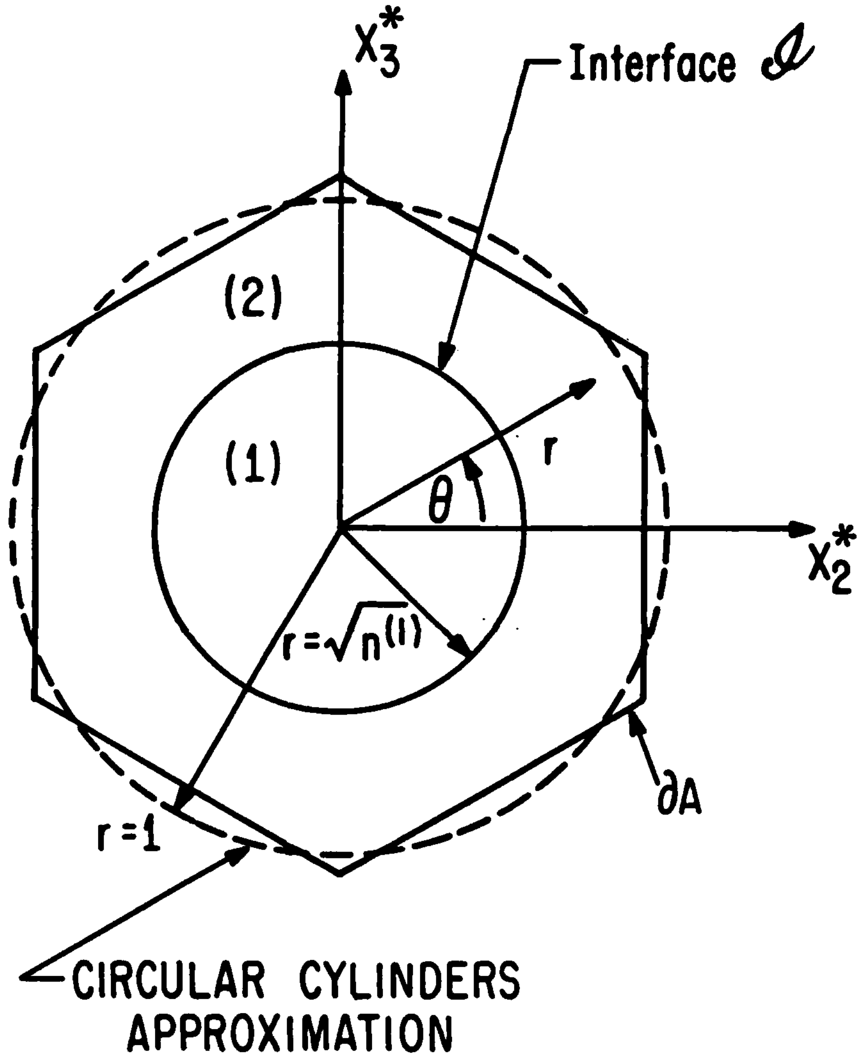

Figure 2. A typical cell

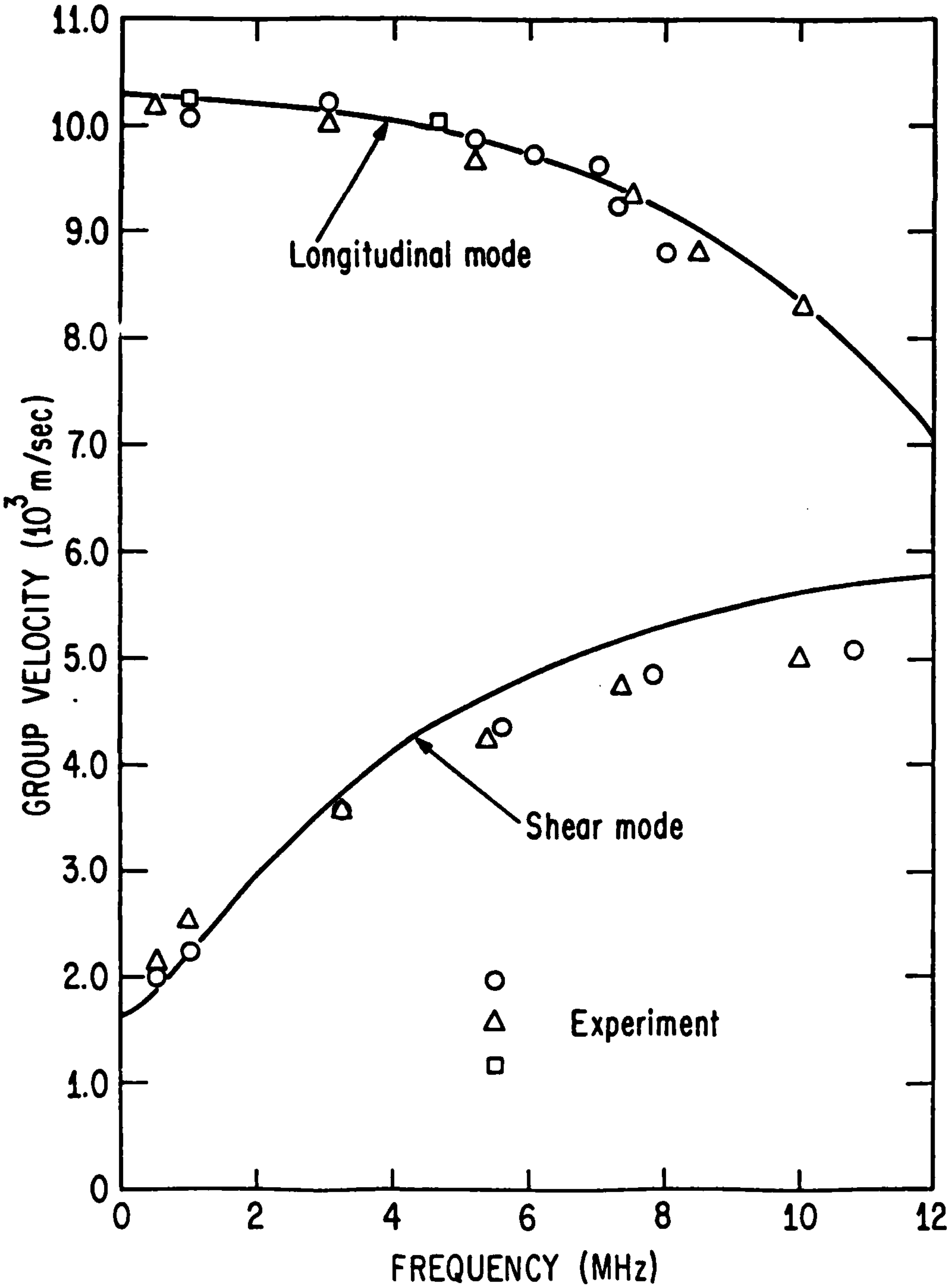

Figure 3. Group velocity spectra of waveguide modes for a boron/epoxy composite(Tauchert and Guzelse, 1972)

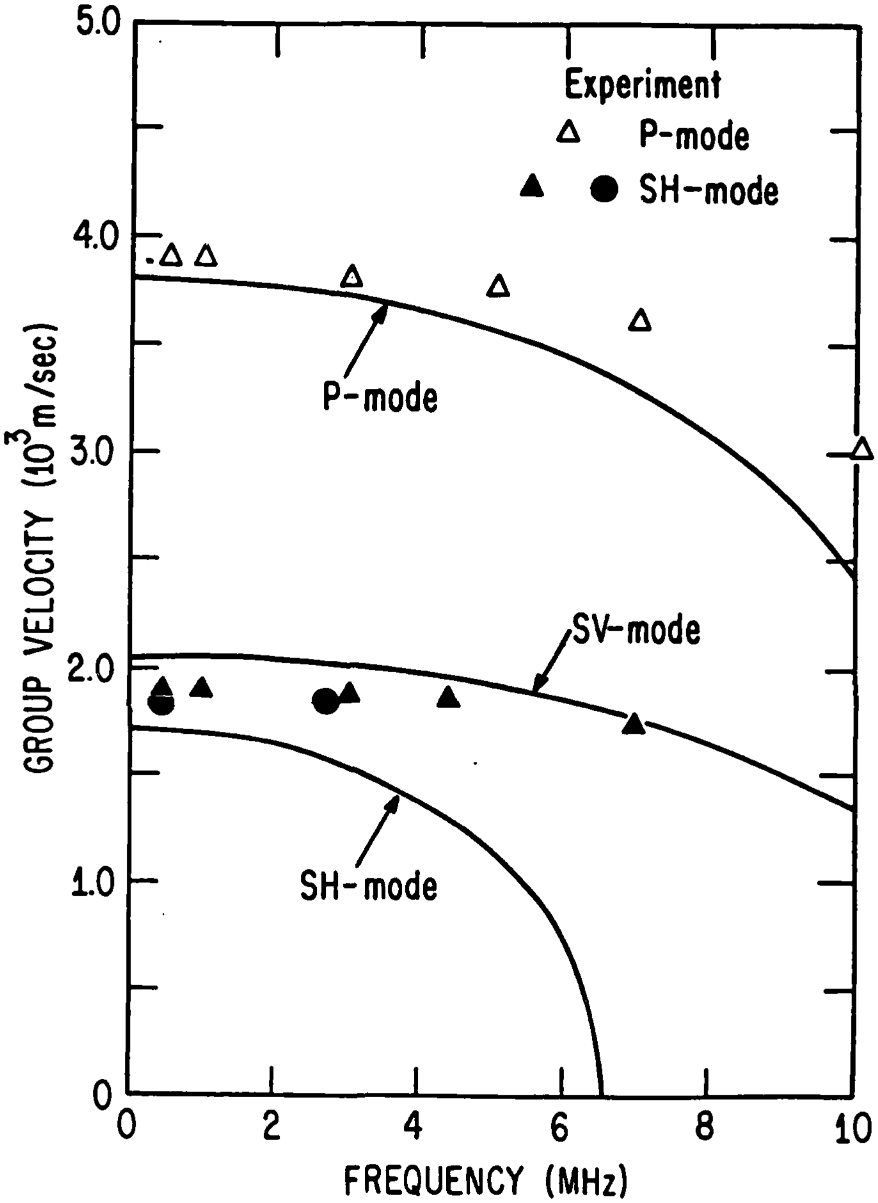

Figure 4. Group velocity spectra of wavereflect modes for a boron/epoxy composite(Tauchert and Guzelse,1972)

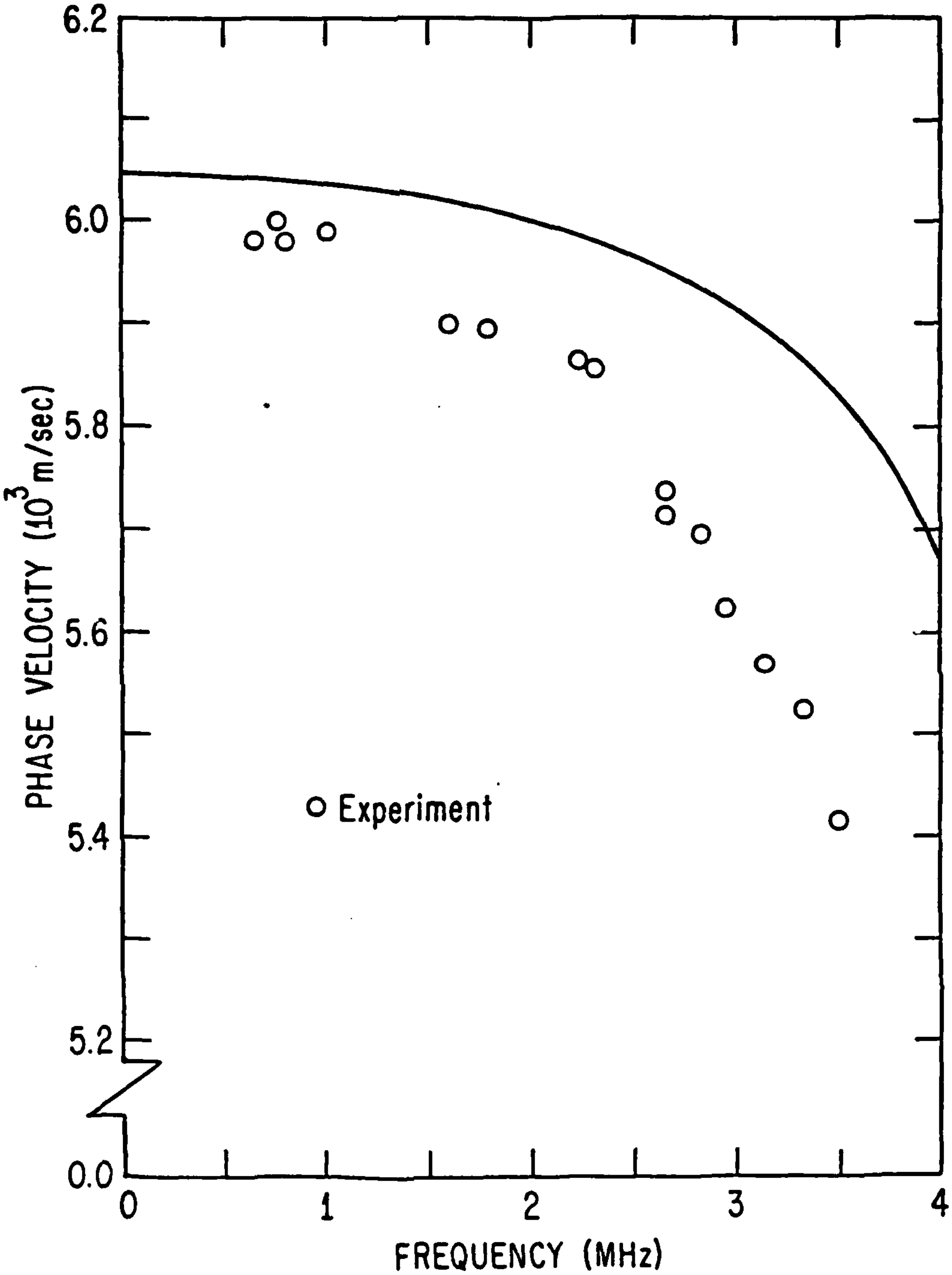

Figure 5. A phase velocity spectrum of a longitudinal wavereflect mode for a tungusten/aluminum composite (Sutherland and Lingle,1972)

APPENDIX B

LAMINATED COMPOSITE PLATE THEORY WITH IMPROVED IN-PLANE RESPONSES

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ABSTRACT

In order to improve the accuracy of the in-plane response of the shear deformable laminated composite plate theory, a new laminated plate theory has been developed based upon a new variational principle proposed by Reissner (1984). The improvement is achieved by including a zigzag shaped C^0 function to approximate the thickness variation of in-plane displacements. The accuracy of this theory is examined by applying it to a problem of cylindrical bending of laminated plates which has been solved exactly by Pagano (1970). The comparison of the in-plane response with the exact solutions for symmetric 3-ply and 5-ply layers has demonstrated that the new theory predicts the in-plane response very accurately even for small span-to-depth ratios.

INTRODUCTION

The advent of metal matrix composites and their application in the form of laminated plates has created a demand for the development of a laminated composite plate theory in which each layer may experience plastic deformation with constraint hardening, (Dvorak and coworkers, 1976, 1984), and transverse cracking. In order to simulate the inelastic responses of each layer, it can be assumed that in-plane strains are primary quantities, while transverse strains are secondary quantities.

In his series of papers Pagano (1970, 1972) obtained elasticity solutions for bidirectional composites for the problem of cylindrical bending and for the problem of the simply supported rectangular plate. Pagano showed the importance of the transverse shear effect for the prediction of accurate plate deflections and the necessity of improving assumptions for in-plane displacements, which are assumed to be linear across the thickness of the plate in the Kirchhoff as well as the Reissner-Mindlin plate theories. Since the development of laminated plate theories including the effect of the transverse shear by Yang, Norris and Stavsky (1966), and Whitney and Pagano (1970) many higher order laminated plate theories have been proposed. Historical accounts of such efforts may be found in the articles by Seide (1980), Bert (1984) and Reddy (1984). However, only a few attempts have been made to improve the in-plane strain response.

It has been stated by Whitney (1972) that the accuracy of classical laminated plate theories for determining in-plane stresses is not improved by the inclusion of shear deformation. Among continuum plate theories in which the number of equilibrium equations does not increase with the number of layers, an improvement of the in-plane response was attempted by including the cubic variation of the in-plane displacements across the thickness of the plate (Hilderbrand, Reissner and Thomas, 1949, and Lo, Christensen and Wu, 1977). An *a posterior* estimate proposed by Whitney (1972) uses the transverse shear strain obtained by the first-order shear deformation theory, (Whitney and Pagano, 1970), to obtain in-plane displacements by integration. This estimate yields an accurate prediction of the in-plane strains. However, the drawback of the approach is that the stress fields do not satisfy plate equilibrium equations.

In order to facilitate a theory which accurately predicts in-plane response, a new laminated plate theory has been developed with the help of a new variational principle, (Reissner, 1984). The improvement is achieved by including a zigzag-shaped C^0 function to approximate the thickness variation of the in-plane displacements. The advantage of using the new Reissner variational principle is that it automatically yields the transverse shear constitutive relations with appropriate shear correction factors. A comparison of the in-plane displacements and stresses predicted by the proposed theory, with Pagano's exact solution of laminated plates in cylindrical bending indicates that inclusion of the zigzag function predicts the in-plane responses more efficiently than the inclusion of smooth nonlinear functions.

FORMULATION

Consider a laminated composite plate composed of N orthotropic layers whose principal axes coincide with rectangular Cartesian coordinates x_1, x_2 and x_3. The coordinate system is selected with x_3 normal to

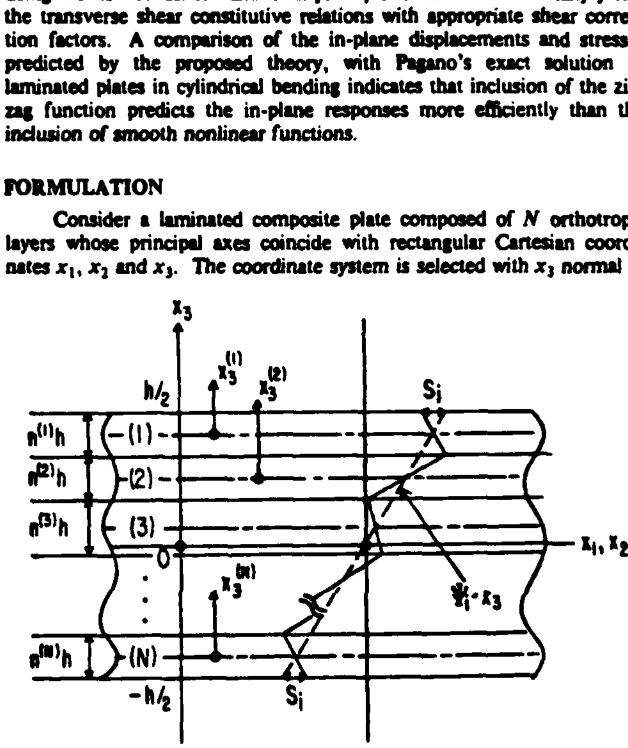


Fig. 1 Coordinate system and the approximation of in-plane displacements

the plane defined by x_1 and x_2. For notational convenience, $(\ )^{(k)}$, $k = 1,2, \cdots ,N$ will denote quantities associated with the k-th layer. For a plate of total thickness h, the thickness of the k-th layer is denoted by $n^{(k)}h$ as shown in Fig. 1, in which the volume fraction $n^{(k)}$ satisfies

$$\sum_{k=1}^{N} n^{(k)} = 1 \ . \tag{1}$$

Unless otherwise stated, the usual Cartesian indicial notation will be employed where Latin indices range from 1 to 3 and repeated indices imply the summation convention. In addition, the notation $(\ )_{,j} \equiv \partial(\ )/\partial x_j$ will be employed.

With the aid of the foregoing notation the governing equations for the displacement vector $u_i^{(k)}$ and the stress tensor $\sigma_{ij}^{(k)}$ in the k-th layer become:

(a) Equilibrium equations

$$\sigma_{ji,j}^{(k)} + f_i^{(k)} = 0 \ , \ \sigma_{ij}^{(k)} = \sigma_{ji}^{(k)} \tag{2}$$

where f_i is body force;

(b) Constitutive equations for orthotropic layers

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} C_{13} \\ C_{23} \\ 0 \end{bmatrix}^{(k)} \left[\frac{\sigma_{33}}{C_{33}} \right]^{(k)} \tag{3a}$$

$$\begin{bmatrix} e_{33} \\ 2e_{23} \\ 2e_{31} \end{bmatrix}^{(k)} = \frac{-1}{C_{33}^{(k)}} \begin{bmatrix} C_{13} & C_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} \frac{1}{C_{33}} & 0 & 0 \\ 0 & \frac{1}{C_{44}} & 0 \\ 0 & 0 & \frac{1}{C_{55}} \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)} \tag{3b}$$

where $\tilde{C}_{ij}^{(k)}$ and $C_{ij}^{(k)}$ are elastic moduli in which $\tilde{C}_{11}$, $\tilde{C}_{12}$ and $\tilde{C}_{22}$ are the reduced stiffness of Whitney and Pagano (1970);

(c) Strain-Displacement Relations

$$e_{ij}^{(k)} = (u_{i,j}^{(k)} + u_{j,i}^{(k)})/2 \ ; \tag{4}$$

(d) Interface Continuity Conditions:

$$u_i^{(k)} = u_i^{(k+1)} \ , \ \sigma_{3i}^{(k)} = \sigma_{3i}^{(k+1)} \ \ k = 1,2,\ldots,N-1 \ ; \tag{5}$$

(e) Upper and Lower Surface Stress Conditions

$$\sigma_{3i}^{(1)} = T_i^{+} \qquad \text{on} \quad x_3 = h/2 \tag{6a}$$

$$\sigma_{3i}^{(N)} = T_i^{-} \qquad \text{on} \quad x_3 = -h/2 \ . \tag{6b}$$

The objective of the subsequent analysis is to derive a laminated plate theory which would have improved in-plane normal strain approximation across the thickness and also the effect of transverse shear deformation.

The basis of a new laminated plate theory is facilitated by a new mixed variational principle, (Reissner, 1984), applied to the N-layered composite plate whose middle plane occupies a domain D in the x_1, x_2-plane.

$$\int\!\!\int_D \Bigg[\sum_{k=1}^{N} \int_{A^{(k)}} \Big\{ \delta e_{11}^{(k)} \sigma_{11}^{(k)} + \delta e_{22}^{(k)} \sigma_{22}^{(k)} + 2\delta e_{12}^{(k)} \sigma_{12}^{(k)}$$

$$+ \delta e_{33}^{(k)} \tau_{33}^{(k)} + 2\delta e_{23}^{(k)} \tau_{23}^{(k)} + 2\delta e_{31}^{(k)} \tau_{31}^{(k)}$$

$$+ \delta \tau_{33}^{(k)} \left[u_{3,3}^{(k)} - e_{33}^{(k)}(\cdots) \right] + \delta \tau_{23}^{(k)} \left[u_{2,3}^{(k)} + u_{3,2}^{(k)} - 2e_{23}^{(k)}(\cdots) \right]$$

$$+ \delta \tau_{31}^{(k)} \left[u_{3,1}^{(k)} + u_{1,3}^{(k)} - 2e_{31}^{(k)}(\cdots) \right] \Big\} dx_3 \Bigg] dx_1 dx_2$$

$$- \int\!\!\int_D \Bigg[\sum_{k=1}^{N} \int_{A^{(k)}} \delta u_i^{(k)} f_i^{(k)} dx_3 + \delta u_i (x_1,x_2,\frac{h}{2}) T_i^{+}$$

$$- \delta u_i (x_1,x_2,-\frac{h}{2}) T_i^{-} \Bigg] dx_1 dx_2$$

$$+ \int_{\partial D_T} \left\{ \sum_{k=1}^{N} \int_{A^{(k)}} \delta u_i^{(k)} \check{T}_i \, dx_3 \right\} dS \tag{7}$$

where ∂D_T denotes the boundary of D with the outward normal ν_i, on which tractions $\check{T}_i$ are specified, and $A^{(k)}$ is the x_3-domain occupied by the k-th layer. In (7) $e_{3i}(\cdots)$ implies the appropriate right hand side of (3b). For an arbitrary variation of u_i and τ_{3i} which is the approximation of σ_{3i}, which satisfy (5) and (6), it can easily be shown that equations (2a) and (3b) are obtained as the Euler-Lagrange equations of (7). In the process, equations (3a) and (4) are considered to be the definitions of $e_{ij}^{(k)}$, $\sigma_{11}^{(k)}$, $\sigma_{22}^{(k)}$, and $\sigma_{12}^{(k)}$.

TRIAL DISPLACEMENTS AND TRANSVERSE STRESSES

A new laminated plate theory which includes the effect of transverse shear and the improved in-plane normal strain approximation is facilitated by introducing appropriate trial displacement and transverse stress fields into the Reissner variational equation (7). As a trial displacement, a zigzag in-plane displacement variation across the thickness of the plate, whose amplitude is expressed by $S_i(x_1,x_2)$, has been included in addition to the linear variation of the Reissner-Mindlin plate theory, as depicted in Fig. 1:

$$u_i^{(k)}(x_1,x_2,x_3) = U_i(x_1,x_2) + \Psi_i(x_1,x_2)x_3$$

$$+ S_i(x_1,x_2)(-1)^k \frac{2}{n^{(k)}h} x_3^{(k)} \ \ i = 1,2 \tag{8a}$$

$$u_3^{(k)}(x_1,x_2,x_3) = U_3(x_1,x_2) \tag{8b}$$

where

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)} \tag{9}$$

is a local x_3-coordinate system with its origin at the center of the k-th layer: $x_{30}^{(k)}$. The inclusion of the zigzag function was motivated by the displacement microstructure of laminated composites (Murakami, Maewal and Hegemier, 1981). In order to test the effect of in-plane zigzag displacement without additional complications, the trial functions (8) are chosen to yield the lowest order theory. As a possible higher order theory one may add the zigzag functions to the displacement trial functions of Lo, Christensen and Wu (1977).

Transverse shear stresses are approximated by

$$\tau_{3i}^{(k)} = Q_i^{(k)}(x_1,x_2)q^{(k)}(x_3) + T_i^{(k-1)}(x_1,x_2)p_u^{(k)}(x_3)$$

$$+ T_i^{(k)}(x_1,x_2)p_l^{(k)}(x_3) \ , \ i = 1,2 \tag{10a}$$

$$\tau_{33}^{(k)} = 0 \tag{10b}$$

where

$$q^{(k)}(x_3^k) = \frac{3}{2n^{(k)}h} \left\{ 1 - \left[\frac{2x_3^{(k)}}{n^{(k)}h} \right]^2 \right\} \ ,$$

$$p_u^{(k)}(x_3^{(k)}) = 3 \left[\frac{x_3^{(k)}}{n^{(k)}h} \right]^2 + \frac{x_3^{(k)}}{n^{(k)}h} - \frac{1}{4} \ ,$$

$$p_i^{(k)}(x_3^{(k)}) = 3\left\{\frac{x_3^{(k)}}{n^{(k)}h}\right\}^2 - \frac{x_3^{(k)}}{n^{(k)}h} - \frac{1}{4} \tag{11}$$

In (10) $T_i^{(k-1)}$ and $T_i^{(k)}$ are the values of τ_{3i} at the upper and lower faces of the k-th layer, respectively, and

$$Q_i^{(k)} = \int_{A^{(k)}} \tau_{3i}^{(k)}\, dx_3 \ . \tag{12}$$

From the definitions of $T_i^{(k)}$ and (6) one obtains

$$T_i^{(o)} = T_i^+ \ , \ T_i^{(N)} = T_i^- \ . \tag{13}$$

Due to the approximation for $u_3^{(k)}$ which yields $e_{33}^{(k)} = 0$, σ_{33} becomes a reactive stress. As a result, σ_{33} is obtained by integrating the x_3-equilibrium equation.

LAMINATED PLATE EQUATIONS

By substituting (8) and (10) into (7) with (3a) and (4), and integrating by parts, one obtains:

(a) Equilibrium equations

$$N_{1i,1} + N_{2i,2} + (T_i^+ - T_i^-) + F_i^o = 0 \ , \ i = 1,2,3 \tag{14}$$

$$M_{1i,1} + M_{2i,2} - N_{3i} + h(T_i^+ + T_i^-)/2 + F_i^I = 0 \ , \ i = 1,2 \tag{15}$$

$$L_{1i,1} + L_{2i,1} - K_{3i} - (T_i^+ - (-1)^N T_i^-) + F_i^{II} = 0 \ , \ i = 1,2 \tag{16}$$

where

$$(N_{ij}, M_{ij}, L_{ij}) \equiv \sum_{k=1}^{N} \int_{A^{(k)}} (1, x_3 \ ,$$

$$(-1)^k \frac{2}{n^{(k)}h} x_3^{(k)})\sigma_{ij}^{(k)}\, dx_3 \ , \ i,j = 1,2 \tag{17a}$$

$$(N_{3i}, K_{3i}) \equiv \sum_{k=1}^{N} \int_{A^{(k)}} (1, (-1)^k \frac{2}{n^{(k)}h})\tau_{3i}^{(k)}\, dx_3 \ , \tag{17b}$$

$$(F_i^o, F_i^I, F_i^{II}) = \sum_{k=1}^{N} \int_{A^{(k)}} \left\{1, x_3 \ , \right.$$

$$\left.(-1)^k \frac{2}{n^{(k)}h\, x_3^{(k)}}\right\} f_i^{(k)}\, dx_3 \ , \ i = 1,2,3 \ ; \tag{17c}$$

(b) Constitutive relations

$$Q_i^{(k)} - \frac{n^{(k)}h}{12}(T_i^{(k-1)} + T_i^{(k)})$$

$$= \hat{C}_i^{(k)} \left\{\frac{5}{6} n^{(k)}h\,(U_{3,i} + \Psi_i) + (-1)^k \frac{5}{3} S_i\right\} \ , \ i = 1,2, \ k = 1,2,\ldots,N \tag{18}$$

$$\frac{2}{15} h \left\{ -\frac{1}{4} \frac{n^{(k)}}{\hat{C}_i^{(k)}} T_i^{(k-1)} + \left[\frac{n^{(k)}}{\hat{C}_i^{(k)}} + \frac{n^{(k+1)}}{\hat{C}_i^{(k+1)}}\right] T_i^{(k)} \right.$$

$$\left. - \frac{1}{4} \frac{n^{(k+1)}}{\hat{C}_i^{(k+1)}} T_i^{(k+1)} \right\}$$

$$= \frac{1}{10}\left\{Q_i^{(k)}/\hat{C}^{(k)} + Q_i^{(k+1)}/\hat{C}^{(k+1)}\right\} \ ,$$

$$i = 1,2 \ , \ k = 1, \cdots, N-1 \tag{19}$$

where, in (18) and (19), no sum is implied on i, and

$$\hat{C}_i^{(k)} \equiv (\delta_{i1}\, C_{55}^{(k)} + \delta_{i2} C_{44}^{(k)}) \ . \tag{20}$$

$Q_i^{(k)}$ and $T_i^{(k)}$ can be obtained in terms of $U_{3,i} + \Psi_i$ and S_i by (18), (19) and (13). As a result, N_{3i} and K_{3i} are expressed by $U_{3,i} + \Psi_i$ and S_i through (17b); such expressions automatically include the appropriate shear correction factors by virtue of the Reissner variational principle.

The appropriate boundary conditions to the equilibrium equations (14-16) are:

$$U_i \ \text{or} \ N_{1i}\nu_1 + N_{2i}\nu_2 \ , \ i = 1,2,3 \tag{21a}$$

$$\Psi_i \ \text{or} \ M_{1i}\nu_1 + M_{2i}\nu_2 \ , \ i = 1,2 \tag{21b}$$

$$S_i \ \text{or} \ L_{1i}\nu_1 + L_{2i}\nu_2 \ , \ i = 1,2 \ . \tag{21c}$$

The remaining constitutive relations for N_{ij}, M_{ij} and L_{ij}, $i,j = 1,2$ are obtained from (17a), (3a), (4), (8) and (10b); the results are:

$$\begin{bmatrix} N_{11} \\ N_{22} \end{bmatrix} = \sum_{k=1}^{N} n^{(k)}h \begin{bmatrix} \hat{C}_{11} \hat{C}_{12} \\ \hat{C}_{12} \hat{C}_{22} \end{bmatrix}^{(k)}$$

$$\times \left\{\begin{bmatrix} U_{1,1} \\ U_{2,2} \end{bmatrix} + x_{30}^{(k)} \begin{bmatrix} \Psi_{1,1} \\ \Psi_{2,2} \end{bmatrix}\right\} \ ,$$

$$N_{12} = \sum_{k=1}^{N} n^{(k)}h\, C_{66}^{(k)} \left\{(U_{1,2} + U_{2,1}) \right.$$

$$\left. + x_{30}^{(k)}\,(\Psi_{1,2} + \Psi_{2,1})\right\} \ , \tag{22}$$

$$\begin{bmatrix} M_{11} \\ M_{22} \end{bmatrix} = \sum_{k=1}^{N} n^{(k)}h \begin{bmatrix} \hat{C}_{11} \hat{C}_{12} \\ \hat{C}_{12} \hat{C}_{22} \end{bmatrix}^{(k)} \left\{ x_{30}^{(k)} \begin{bmatrix} U_{1,1} \\ U_{2,2} \end{bmatrix} \right.$$

$$+ \left[x_{30}^{(k)2} + \frac{(n^{(k)}h)^2}{12}\right] \begin{bmatrix} \Psi_{1,1} \\ \Psi_{2,2} \end{bmatrix}$$

$$\left. + (-1)^k \frac{n^{(k)}h}{6} \begin{bmatrix} S_{1,1} \\ S_{2,2} \end{bmatrix}\right\} \ ,$$

$$M_{12} = \sum_{k=1}^{N} n^{(k)}h\, C_{66}^{(k)} \left\{x_{30}^{(k)}(U_{1,2} + U_{2,1}) \right.$$

$$+ (x_{30}^{(k)2} + \frac{(n^{(k)}h)^2}{12})(\Psi_{1,2} + \Psi_{2,1})$$

$$\left. + (-1)^k \frac{n^{(k)}h}{6}(S_{1,2} + S_{2,1})\right\} \ , \tag{23}$$

$$\begin{bmatrix} L_{11} \\ L_{22} \end{bmatrix} = \sum_{k=1}^{N} \frac{n^{(k)}h}{3} \begin{bmatrix} \hat{C}_{11} \hat{C}_{12} \\ \hat{C}_{12} \hat{C}_{22} \end{bmatrix}^{(k)}$$

$$\times \left\{(-1)^k \frac{n^{(k)}h}{2} \begin{bmatrix} \Psi_{1,1} \\ \Psi_{2,2} \end{bmatrix} + \begin{bmatrix} S_{1,1} \\ S_{2,2} \end{bmatrix}\right\} \ ,$$

$$L_{12} = \sum_{k=1}^{N} \frac{n^{(k)}h}{3} C_{66}^{(k)} \left\{(-1)^k \frac{n^{(k)}h}{2}(\Psi_{1,2} + \Psi_{2,1}) \right.$$

$$\left. + (S_{1,2} + S_{2,1})\right\} \tag{24}$$

CYLINDRICAL BENDING OF LAMINATED PLATES

In order to examine the accuracy of the new laminated plate theory cylindrical bending of bidirectional plates is considered. The plate is simply supported on the ends $x_l = 0$ and l, and is infinitely long in the x_2-direction. The prescribed boundary conditions on the upper and lower faces are given by

$$T_3^+ = q_o \sin\frac{\pi}{l} x_1 \,,\; T_1^+ = T_1^- = T_3^- = 0 \,. \tag{25}$$

The boundary conditions for the simply supported ends are from (21):

$$N_{11} = U_3 = M_{11} = L_{11} = 0 \quad \text{at } x_1 = 0, l \,. \tag{26}$$

For the clarity of presentation the constitutive relations (22-24) are cast in a nondimensional form, in which h and reference moduli E_T are used:

$$\begin{bmatrix} N_{11}/(hE_T) \\ M_{11}/(h^2E_T) \\ L_{11}/(hE_T) \end{bmatrix} = \begin{bmatrix} A_{11} & B_{12} & 0 \\ B_{12} & C_{11} & C_{12} \\ 0 & C_{12} & C_{22} \end{bmatrix} \begin{bmatrix} U_{1,1} \\ h\Psi_{1,1} \\ S_{1,1} \end{bmatrix} \tag{27}$$

where

$$A_{11} = \sum_{k=1}^{N} n^{(k)} (\check{C}_{11}^{(k)}/E_T) \,,\; B_{12} = \sum_k n^{(k)} (x_{30}^{(k)}/h)(\check{C}_{11}^{(k)}/E_T) \,,$$

$$C_{11} = \sum_k n^{(k)} \{(x_{30}^{(k)}/h)^2 + n^{(k)2}/12\} (\check{C}_{11}^{(k)}/E_T) \,,$$

$$C_{12} = \sum_k (-1)^k (n^{(k)2}/6)(\check{C}_{11}^{(k)}/E_T) \,,\; C_{22} = \sum_k (n^{(k)}/3)(C_{11}^{(k)}/E_T) \tag{28}$$

Equations (18) and (19) may be expressed in a matrix form, respectively:

$$\underline{Q} - [A]\, \underline{T} = \underline{d}_1 (U_{3,1} + \Psi_1) + \underline{d}_2 (S_1/h) \tag{29}$$

$$[B]\underline{T} = [F]\, \underline{Q} \tag{30}$$

where

$$\underline{Q} = [Q_1^{(1)}, Q_1^{(2)}, \cdots, Q_1^{(N)}]^T/(hE_T)$$

$$\underline{T} = [T_1^{(1)}, T_1^{(2)}, \cdots, T_1^{(N-1)}]^T/E_T \tag{31}$$

and $[A]$, $[B]$ and $[F]$ are matrices of $N \times (N-1)$, $(N-1) \times (N-1)$ and $(N-1) \times N$, respectively. In (31) $[\;]^T$ implies the transpose of $[\;]$. Solving (29) and (30) for $\underline{Q}$ one obtains

$$\underline{T} = [B]^{-1} [F]\, \underline{Q} \tag{32}$$

$$\underline{Q} = \underline{d}(U_{3,1} + \Psi_1) + \underline{d}_2(S_1/h) \tag{33}$$

where

$$(\underline{d}_1, \underline{d}_2) = ([I] - [A][B]^{-1}[F])^{-1} (\underline{d}_1, \underline{d}_2) \tag{34}$$

and $[I]$ is an $N \times N$ identity matrix. Finally substituting (33) into (17b), one obtains

$$\begin{bmatrix} N_{31}/(hE_T) \\ K_{31}/E_T \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} U_{3,1} + \Psi_1 \\ S_1/h \end{bmatrix} \tag{35}$$

where

$$D_{11} = \underline{e}_1^T \cdot \underline{d}_1 \,,\; D_{12} = \underline{e}_1^T \cdot \underline{d}_2$$

$$D_{21} = \underline{e}_2^T \cdot \underline{d}_1 \,,\; D_{22} = \underline{e}_2^T \cdot \underline{d}_2 \tag{36}$$

and

$$\underline{e}_1 = [1, 1, \ldots, 1]^T$$

$$\underline{e}_2 = 2[-1/n^{(1)}, 1/n^{(2)}, \ldots, (-1)^N/n^{(N)}]^T \tag{37}$$

are $N \times 1$ matrices.

For the cylindrical bending problem equations (14)-(16) reduce to

$$N_{11,1} = 0 \tag{38a}$$

$$N_{13,1} + q_o \sin\frac{\pi}{l} x_1 = 0 \tag{38b}$$

$$M_{11,1} - N_{31} = 0 \tag{38c}$$

$$L_{11,1} - K_{31} = 0 \tag{38d}$$

From the boundary condition (26) for N_{11} one concludes that

$$N_{11} = 0 \quad \text{for } 0 \leqslant x \leqslant l \,. \tag{39}$$

By using (39) one can rewrite (27) as follows:

$$\begin{bmatrix} M_{11}/(h^2E_T) \\ L_{11}/(hE_T) \end{bmatrix} = \begin{bmatrix} C_{11} - B_{12}^2/A_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \begin{bmatrix} h\Psi_{1,1} \\ S_{1,1} \end{bmatrix} \tag{40}$$

The constitutive relations (40), (35) and the form of forcing terms in (38) suggest the following form of the displacements:

$$U_3/h = \hat{u}_3 \sin(\pi/l)x_1 \,,\; \tilde{\Psi}_1 = \phi_1 \cos(\pi/l)x_1$$

$$S_1/h = s_1 \cos(\pi/l)x_1 \tag{41}$$

It can be easily shown that equations (41) satisfy boundary conditions (26). The equations for the unknown constants $\hat{u}_3$, ϕ_1 and s_1 are obtained by substituting (41), (40), (35) into (38b-d). The result is

$$\begin{bmatrix} p^2 D_{11} & p D_{11} & p D_{12} \\ p D_{11} & D_{11} + p^2(C_{11} - B_{12}^2/A_{11}) & D_{12} + p^2 C_{12} \\ p D_{21} & D_{21} + p^2 C_{12} & D_{22} + p^2 C_{22} \end{bmatrix} \begin{bmatrix} \hat{u}_3 \\ \phi_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} q_o/E_T \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

where

$$p \equiv \pi h/l \,. \tag{42b}$$

NUMERICAL RESULTS

The accuracy of the proposed theory is assessed by considering the cylindrical bending of a symmetric three-ply laminate for which an exact solution has been obtained by Pagano (1970). The material properties are for 0 degree layers

$$\check{C}_{11}^{(1)}/E_T = \check{C}_{11}^{(3)}/E_T = 25.0627$$

$$C_{33}^{(1)}/E_T = C_{33}^{(3)}/E_T = 0.5 \tag{43a}$$

and for 90 degree layers

$$\check{C}_{11}^{(2)}/E_T = 1.0025 \,,\; C_{33}^{(2)}/E_T = 0.2 \,. \tag{43b}$$

In the following figures the results are presented in terms of the nondimensional quantities adopted by Pagano (1970):

$$\bar{u}_1 = \frac{E_T\, u_1(0, x_3)}{hq_o} \,,\; \bar{u}_3 = \frac{100 E_T h^3}{q_o l^4} u_3 (\frac{l}{2}, 0) \,,$$

$$\bar{\sigma}_{11} = \frac{1}{q_o} \sigma_{11} (\frac{l}{2}, x_3) \,,\; \bar{\sigma}_{13} = \frac{1}{q_o} \sigma_{13} (0, x_3) \,,$$

$$\bar{x}_3 = x_3/h \,. \tag{44}$$

In the various curves, the solid line indicates the elasticity solution (Exact), while the results by the proposed theory and the first order shear deformation theory (FSD) are shown by a broken line and a dashed-dotted line, respectively.

For a symmetric three-ply laminate (0/90/0) with layers of equal thickness, the plate deflection - the span-to-depth ratio relations have been plotted in Fig. 2. The results of the first order shear deformation theory have been computed by setting $s_1 = 0$ in (42). Since the present theory is a higher order theory than the first order shear deformation theory, the present theory predicts $\bar{u}_3$ more accurately than the FSD. The variation of the in-plane displacement $\bar{u}_1$ across the thickness of the

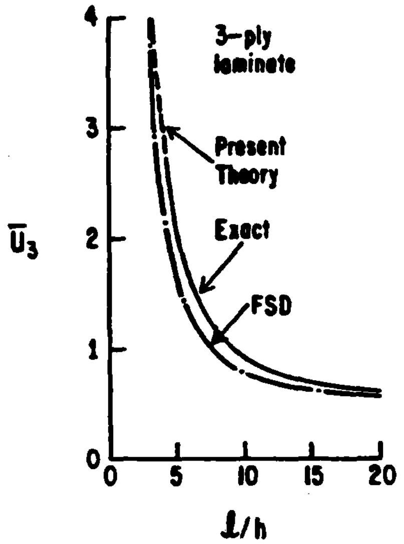


Fig. 2 Comparison of mid-surface transverse displacements

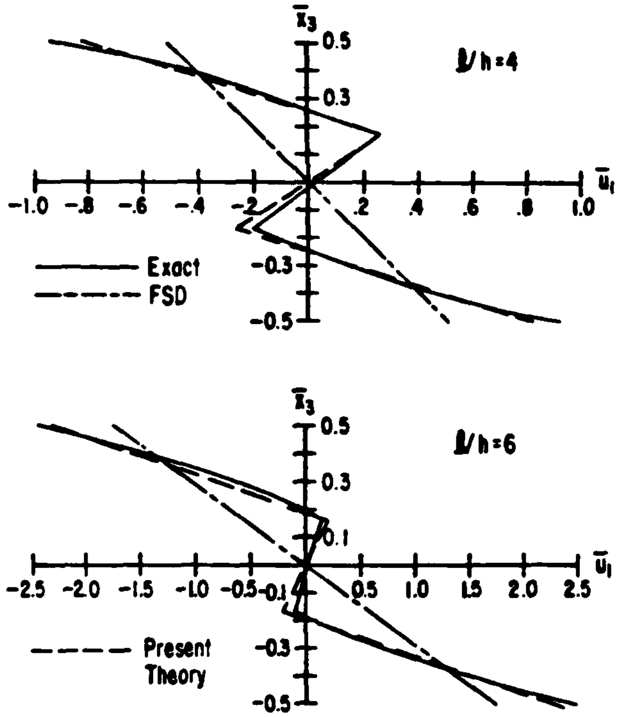
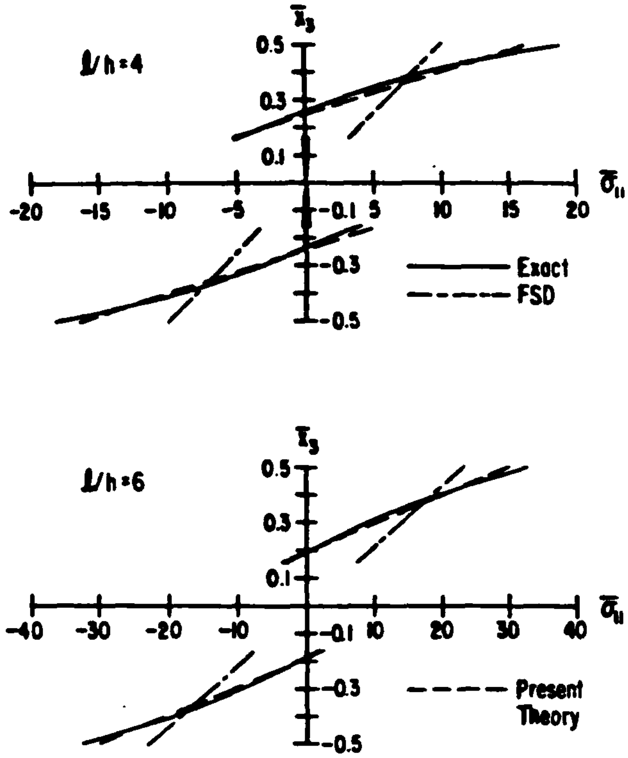


Fig. 3 In-plane displacement variation through the thickness for $l/h = 4$ and 6



Figs. 4 In-plane normal stress variation through the thickness for $l/h = 4$ and 6

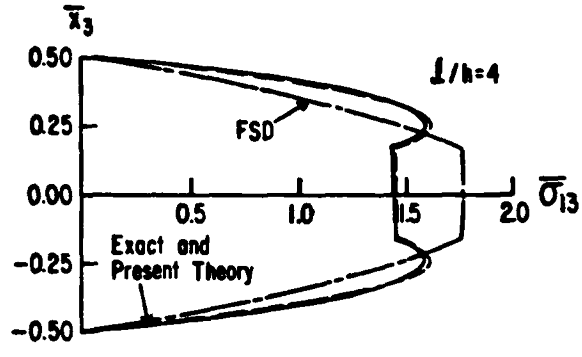


Fig. 5 Transverse shear variation through the thickness for $l/h = 4$

plate has been compared in Figs. 3a and b for $l/h = 4$ and $= 6$, respectively. The results show that the poor approximation of the FSD has been significantly improved by the present theory which includes a zigzag-shaped function for the variation of $\bar{u}_1$ across the thickness. This accuracy is reflected directly in the variation of the in-plane normal stress $\bar{\sigma}_{11}$ as shown in Figs. 4a and b. The variation of the transverse shear stress $\bar{\sigma}_{13}$ for $l/h = 4$ has been shown in Fig. 5, in which the curves for the present theory and the exact solution are almost identical.

As a more difficult test for the proposed theory a symmetric five-ply laminate *(0/90/0/90/0) with layers of equal thickness* has been considered. A comparison of the plate deflection is shown in Fig. 6. The corresponding results for the in-plane displacements, the in-plane normal stresses, and the transverse stresses are shown in Fig. 7, Figs. 8, and *Fig. 9, respectively*. The results reveal that the present theory predicts the plate deformation very accurately even for small l/h. Furthermore, it can be seen from Figs. 3 and 7 that the variation of the in-plane displacements across the thickness of the plate is approximated better by the zigzag-type function in (8a) than by x_3^2 or x_3^3 terms.

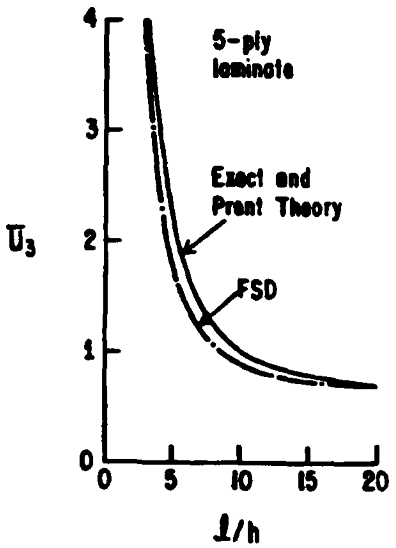
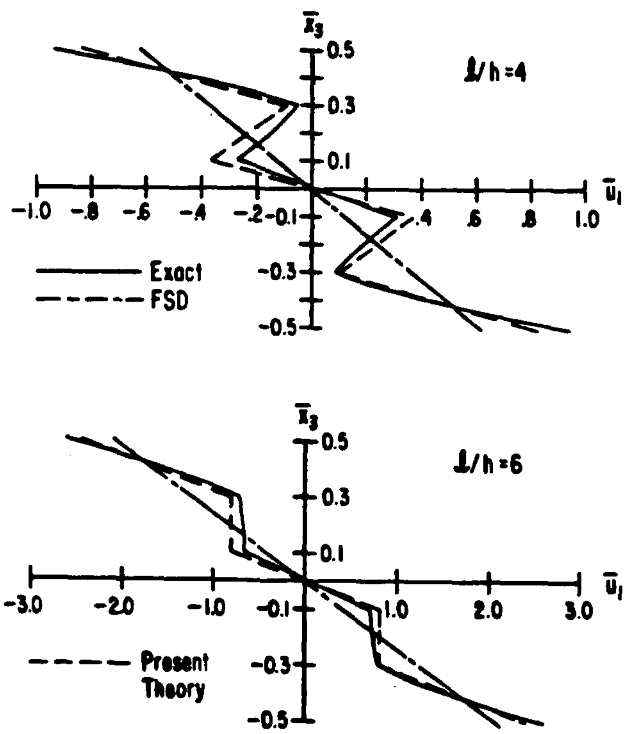
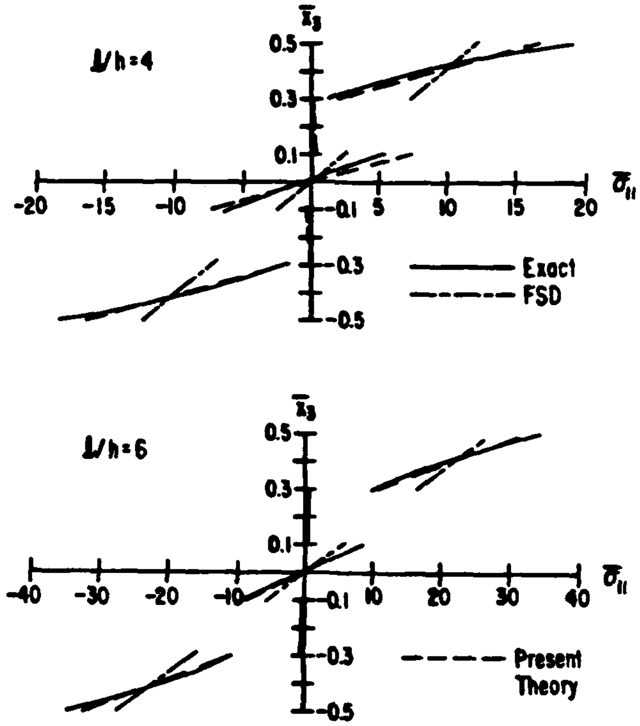


Fig. 6 Comparison of mid-surface transverse displacements



Figs. 7 In-plane displacement variation through the thickness for $l/h = 4$ and 6



Figs. 8 In-plane normal stress variation through the thickness for $l/h = 4$ and 6

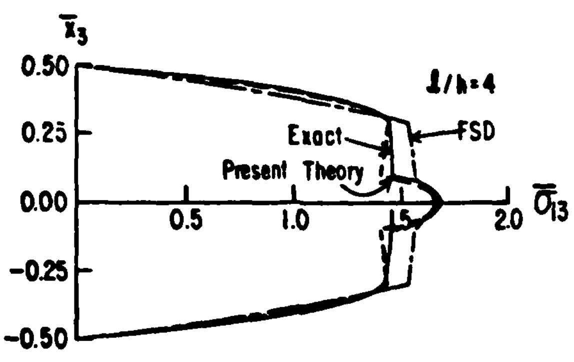


Fig. 9 Transverse shear variation through the thickness for $l/h = 4$

CONCLUSIONS

A laminated composite plate theory which accurately predicts in-plane response has been developed based upon the Reissner new variational principle (1984). A unique aspect of the theory is that it includes a zigzag-shaped C° function to approximate the variation of the in-plane displacement over the thickness. The accuracy of the theory has been examined by considering the cylindrical bending of laminated plates which has been solved exactly by Pagano (1970). The comparison of the plate deflection and the in-plane displacements and normal stresses for symmetric three-ply and five-ply laminates has demonstrated that the new theory predicts the in-plane response very accurately even for small span-to-depth ratios.

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A HIGH-ORDER LAMINATED PLATE THEORY WITH IMPROVED IN-PLANE RESPONSES [1)]

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ABSTRACT

In order to improve the accuracy of the in-plane responses of the shear deformable laminated composite plate theories, a new high-order laminated plate theory was developed based upon Reissner's new mixed Variational Principle [9]. To this end, a zig-zag shaped C^0 function and Legendre polynomials were introduced into the approximate in-plane displacement distributions across the plate thickness. The accuracy of the present theory was examined by applying it to the cylindrical bending problem of laminated plates which had been solved exactly by Pagano [1]. A comparison with the exact solutions obtained for several symmetric and asymmetric cross-ply laminates indicates that the present theory accurately estimates in-plane responses, even for small span-to-thickness ratios.

1. INTRODUCTION

The increasing use of composite materials as thick laminates, in aerospace engineering and in automotive engineering, has clearly demonstrated the need for the development of new theories to efficiently and accurately predict the behavior of such structural components. The intrinsic heterogeneity and anisotropy of these composite structures as evidenced in the stacking of several fibrous layers and in the high discontinuity in material properties across the interfaces, make the classical theories of plates and shells inadequate.

The inspiration and guidelines for the subsequent attempts have stemmed from Pagano's works [1,2,3] where the exact elasticity solutions for the problems of cylindrical bending and simply supported rectangular plates were given. Pagano showed the importance of incorporating the effect of transverse shear deformations in order to accurately estimate the plate lateral deflection and the need to improve upon the thickness variation of the in-plane displacements, which are assumed to be C^1 linear functions in both classical plate theory (CPT) and Reissner-Mindlin plate theory (FSD).

The first attempt to develop a general linear laminated plate theory is credited to Yang, Norris and Stavsky [4]. Their theory is an extension of the Reissner-Mindlin homogeneous plate theory as applied to an arbitrary number of bonded anisotropic layers. Whitney and Pagano [5] extended Yang, Norris and Stavsky's work. An important conclusion drawn from their analysis, which was also emphasized later by Whitney [6], is that the inaccuracies of the classical plate theory at low span-to-thickness ratios for determining in-plane stresses are not alleviated by the introduction of shear deformations. Whitney [6] obtained in-plane displacements by integrating the transverse shear strains deduced in [5]. This resulted in a higher order approximation which accurately predicted in-plane strains, but the resulting modified stresses did not necessarily satisfy the original plate equilibrium equations.

Since then, other high-order laminated plate theories have been proposed that account for transverse shear strains. Of these, the Lo, Christensen and Wu [7] and the Reddy [8] high-order models have served as the foundation for the present theory. In their paper [7], Lo, Christensen and Wu used appropriate higher order terms in the power series expansions of the assumed displacement

field which was proposed by Hildebrand, Reissner and Thomas [8]. On the other hand, Reddy [9] imposed the condition of vanishing transverse shear strains on the top and bottom surfaces of the plate. However, this theory does not satisfy the continuity condition of transverse shear stresses at the interfaces.

The objective of the present paper is to improve the approximation of in-plane variables in laminated plate theories. In-plane displacements and bending and stretching stresses are considered primary quantities in any approximate laminated plate analysis; transverse stresses are only of secondary importance since they are an order of magnitude smaller than the primary bending and stretching stresses. By using a new mixed variational principle proposed by Reissner [10], the present theory is a high-order model which improves upon existing theories by including in the assumed in-plane displacement variations across the plate thickness: 1) a zig-zag shaped C^0 function as detailed by Murakami [11]; and, 2) Legendre polynomials. The advantage of using Reissner's new mixed variational principle is that it automatically yields the appropriate shear correction factors for the transverse shear constitutive equations. Another attractive feature of the proposed theory is that the number of equations to be solved is not increased as the number of layers becomes larger and larger. A comparison of the proposed theory with Pagano's exact elasticity solution for symmetric and asymmetric laminated plates in cylindrical bending, shows that in-plane displacements and stresses are accurately predicted by the inclusion of the zig-zag shaped function and the Legendre Polynomials.

2. FORMULATION

Consider an N-layer laminated composite plate, shown in Fig. 1, with principal axes coinciding with a Cartesian coordinate system (x_1, x_2, x_3), such that the x_3-axis is perpendicular to the plane defined by x_1 and x_2. The following notation: $(\ )^{(k)}$, $k = 1,2,..,N$ will designate quantities associated with the k^{th}-layer. The thickness of each layer is $n^{(k)}h$, where h is the total thickness of the plate. The volume fractions $n^{(k)}$ satisfy the relation

$$\sum_{k=1}^{N} n^{(k)} = 1 \tag{1}$$

Unless otherwise specified, the usual cartesian indicial notation is employed where latin and greek indices range from 1 to 3 and 1 to 2, respectively. Repeated indices imply the summation convention and $(\ )_{,i}$ is used to denote partial differentiation with respect to x_i.

With the help of the foregoing notation, the governing equations for the displacement vector $u_i^{(k)}$ and stress tensor $\sigma_{ij}^{(k)}$ associated with the k^{th}-layer are:

a) Equilibrium Equations

$$\sigma_{ji,j}^{(k)} + f_i^{(k)} = 0 \quad ; \quad \sigma_{ij}^{(k)} = \sigma_{ji}^{(k)} \tag{2}$$

where f_i are the body forces;

b) Constitutive Equations For Orthotropic Layers

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{12} & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} C_{13}/C_{33} \\ C_{23}/C_{33} \\ 0 \end{bmatrix}^{(k)} \sigma_{33}^{(k)} \tag{3a}$$

$$\begin{bmatrix} e_{33} \\ 2e_{23} \\ 2e_{31} \end{bmatrix}^{(k)} = - \begin{bmatrix} C_{13}/C_{33} & C_{23}/C_{33} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} 1/C_{33} & 0 & 0 \\ 0 & 1/C_{44} & 0 \\ 0 & 0 & 1/C_{55} \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)} \tag{3b}$$

where C_{ij} are the elastic constants and $\bar{C}_{ij}$ $(i,j = 1,2,6)$ represent the reduced stiffnesses introduced by Whitney and Pagano [5];

c) Strain-Displacement Relations

$$e_{ij}^{(k)} = \frac{1}{2}\left(u_{i,j}^{(k)} + u_{j,i}^{(k)}\right) \; ; \tag{4}$$

d) Interface Continuity Conditions

$$u_i^{(k)} = u_i^{(k+1)} \;\; , \;\; \sigma_{3i}^{(k)} = \sigma_{3i}^{(k-1)} \;\; ; \;\; k = 1,2,..,N-1 \;\; ; \tag{5}$$

e) Upper and Lower Surface Stress Conditions

$$\sigma_{3i}^{(1)} = T_i^+ \quad \text{on} \;\; x_3 = \frac{h}{2} \tag{6a}$$

$$\sigma_{3i}^{(N)} = T_i^- \quad \text{on} \;\; x_3 = -\frac{h}{2} \; . \tag{6b}$$

The objective in developing a new laminated plate theory is twofold: first, to improve the assumed variation of in-plane displacements through the thickness of the plate and second, to include the effect of transverse shear deformation. In order to carry out this task, Reissner's new mixed variational principle [10] was applied to the N-layer composite plate whose middle surface occupies a domain D in the x_1, x_2-plane:

$$\iint\limits_D \left[\int\limits_{A^{(k)}} \left\{ \delta e_{ij}^{(k)} \sigma_{ij}^{(k)} + [u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - 2e_{3\alpha}^{(k)}(\cdots)]\delta\tau_{3\alpha}^{(k)} + [u_{3,3}^{(k)} - e_{33}^{(k)}(\cdots)]\delta\tau_{33}^{(k)} \right\} dx_3 \right] dx_1\, dx_2$$

$$= \iint\limits_D \left[\sum_k \int\limits_{A^{(k)}} \delta u_i^{(k)} f_i^{(k)}\, dx_3 \right] dx_1\, dx_2 + \int\limits_{\partial D_T} \left[\sum_k \int\limits_{A^{(k)}} \delta u_i^{(k)} \overset{\nu}{T}_i\, dx_3 \right] ds \tag{7}$$

$$+ \iint\limits_D \left[\delta u_i^{(1)}(x_1, x_2, \frac{h}{2}) T_i^+ - \delta u_i^{(N)}(x_1, x_2, -\frac{h}{2})\, T_i^- \right] dx_1\, dx_2$$

where ∂D_T denotes the boundary of D with outward normal ν_α on which tractions $\overset{\nu}{T}_i$ are specified and $A^{(k)}$ represents the x_3-domain occupied by the k^{th}-layer. Also $e_{3i}(...)$ implies the appropriate right-hand side of (3b). Due to the nature of Reissner's mixed variational principle, Eqs. (3a) are taken to be the definitions of $\sigma_{\alpha\beta}^{(k)}$ used in connection with (7).

3. TRIAL DISPLACEMENT FIELD, TRANSVERSE AND NORMAL STRESSES

The high-order laminated plate theory which takes into account the effect of transverse shear strains, is obtained by including the Legendre polynomials of order $n = 1,2,3$ with respect to the x_3-coordinate to a zig-zag in-plane displacement variation of amplitude $S_i(x_1,x_2)$ across the plate thickness.

The appropriate trial functions used in connection with Reissner's mixed variational principle Eq. (7) are taken to be:

a) Trial Displacement Field

$$u_i^{(k)}(x_1,x_2,x_3) = U_i(x_1,x_2) + (\frac{h}{2})\Psi_i(x_1,x_2)P_1(\zeta) + S_i(x_1,x_2)(-1)^k \frac{2}{n^{(k)}h} x_3^{(k)} + (\frac{h}{2})^2\xi_i(x_1,x_2)P_2(\zeta) + (\frac{h}{2})^3\phi_i(x_1,x_2)P_3(\zeta) \tag{8}$$

where $\zeta \equiv \frac{2x_3}{h}$ and $P_n(\zeta)$ are the Legendre Polynomials of order n. It is also understood that $\phi_3 \equiv 0$. $x_3^{(k)}$ is a local x_3-coordinate system with its origin at the center $x_{30}^{(k)}$ of the k^{th}-layer, i.e.

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)} \ . \tag{9}$$

Eq. (8) may be regarded as a superposition of a zig-zag function and the cubic variation as proposed by Lo, Christensen and Wu [7], with the exception that here Legendre polynomials are used instead of single powers in x_3;

b) Trial Transverse and Normal Stresses

$$\tau_{3\alpha}^{(k)}(x_1,x_2,x_3) = Q_\alpha^{(k)}(x_1,x_2)F_1(z) + R_\alpha^{(k)}(x_1,x_2)F_2(z) + J_\alpha^{(k)}(x_1,x_2)F_3(z) + [T_\alpha^{(k-1)}(x_1,x_2) + T_\alpha^{(k)}(x_1,x_2)]F_4(z) + [T_\alpha^{(k-1)}(x_1,x_2) - T_\alpha^{(k)}(x_1,x_2)]F_5(z) \ ; \tag{10a}$$

$$\tau_{33}^{(k)}(x_1,x_2,x_3) = Q_3^{(k)}(x_1,x_2)F_1(z) + R_3^{(k)}(x_1,x_2)F_6(z) + J_3^{(k)}(x_1,x_2)F_3(z) + I_3^{(k)}(x_1,x_2)F_7(z) + [T_3^{(k-1)}(x_1,x_2) + T_3^{(k)}(x_1,x_2)]F_4(z) + [T_3^{(k-1)}(x_1,x_2) - T_3^{(k)}(x_1,x_2)]F_8(z) \tag{10b}$$

where

$$F_1(z) = \frac{5}{n^{(k)}h}\left[21z^4 - \frac{15}{2}z^2 + \frac{9}{16}\right] \ , \ F_2(z) = \frac{-30}{(n^{(k)}h)^2}(4z^3 - z)$$

$$F_3(z) = \frac{-105}{(n^{(k)}h)^3}(20z^4 - 6z^2 + \frac{1}{4}) \quad , \quad F_4(z) = 35z^4 - \frac{15}{2}z^2 + \frac{3}{16}$$

$$F_5(z) = 10z^3 - \frac{3}{2}z \quad , \quad F_6(z) = \frac{105}{(n^{(k)}h)^2}(36z^5 - 14z^3 + \frac{5}{4}z) \tag{11}$$

$$F_7(z) = \frac{-315}{(n^{(k)}h)^4}(112z^5 - 40z^3 + 3z) \quad , \quad F_8(z) = 126z^5 - 35z^3 + \frac{15}{8}z$$

and $\quad z \equiv \dfrac{x_3^{(k)}}{n^{(k)}h} \quad , -\dfrac{1}{2} \leqslant z \leqslant \dfrac{1}{2}$

Also ,

$$(Q_i^{(k)}, R_i^{(k)}, J_i^{(k)}) \equiv \int_{A^{(k)}} (1, x_3^{(k)}, x_3^{(k)2})\, \tau_{3i}^{(k)}\, dx_3 \tag{12a}$$

$$I_3^{(k)} \equiv \int_{A^{(k)}} x_3^{(k)3}\, \tau_{33}^{(k)}\, dx_3 \tag{12b}$$

In (10) $T_i^{(k-1)}$ and $T_i^{(k)}$ are the values of τ_{3i} at the top and bottom surfaces of the k^{th} layer respectively. From (6)

$$T_i^{(0)} = T_i^+ \text{ and } T_i^{(N)} = T_i^- \ . \tag{13}$$

The degree of the polynomials $F_i(z)$, $i=1-8$, appearing in (10a,b) is consistent with the order of truncation in the assumed expansions (8) for the displacement $u_i^{(k)}$.

4. LAMINATED PLATE EQUATIONS

Substituting (8) and (10) into (7), using Gauss' Theorem and the orthogonality property of the Legendre polynomials one obtains:

a) Equilibrium Equations:

$$N_{\alpha i,\alpha} + T_i^+ - T_i^- + F_i^N = 0 \tag{14a}$$

$$M_{\alpha i,\alpha} - N_{3i} + \frac{h}{2}(T_i^+ + T_i^-) + F_i^M = 0 \tag{14b}$$

$$Z_{\alpha i,\alpha} - K_{3i} - (T_i^+ - (-1)^N T_i^-) + F_i^Z = 0 \tag{14c}$$

$$L_{\alpha i,\alpha} - 3M_{3i} + \frac{h^2}{4}(T_i^+ - T_i^-) + F_i^L = 0 \tag{14d}$$

$$P_{\beta\alpha,\beta} - (5L_{3\alpha} + \frac{h^2}{4} N_{3\alpha}) + \frac{h^2}{8}(T_\alpha^+ + T_\alpha^-) + F_\alpha^P = 0 \tag{14e}$$

where

$$\begin{bmatrix} N_{\alpha\beta}, & M_{\alpha\beta}, & Z_{\alpha\beta}, & L_{\alpha\beta}, & P_{\alpha\beta} \\ F_i^N, & F_i^M, & F_i^Z, & F_i^L, & F_i^P \end{bmatrix} \equiv \sum_{k=1}^{N} \int_{A^{(k)}}$$

$$[1, \frac{h}{2} P_1(\zeta), (-1)^k \frac{2x_3^{(k)}}{n^{(k)}h}, (\frac{h}{2})^2 P_2(\zeta), (\frac{h}{2})^3 P_3(\zeta)] \begin{bmatrix} \sigma_{\alpha\beta}^{(k)} \\ f_i^{(k)} \end{bmatrix} dx_3 \tag{15a,b}$$

$$(N_{3i}, M_{3i}, K_{3i}, L_{3i}) \equiv \sum_{k=1}^{N} \int_{A^{(k)}} \left[1, \frac{h}{2} P_1(\zeta), (-1)^k \frac{2}{n^{(k)}h}, (\frac{h}{2})^2 P_2(\zeta)\right] \tau_{3i}^{(k)} dx_3 \; ; \tag{15c}$$

b) Constitutive Equations:

- For Transverse Stresses

$$Q_\alpha^{(k)} - \frac{8 J_\alpha^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{30}\left(T_\alpha^{(k-1)} + T_\alpha^{(k)}\right) = \frac{2}{5} h\, n^{(k)} \bar{C}_\alpha^{(k)} \Big[U_{3,\alpha} + \Psi_\alpha + S_\alpha (-1)^k$$

$$\frac{2}{n^{(k)}h} + h\, n_o^{(k)}(\Psi_{3,\alpha} + 3\xi_\alpha) + \frac{h^2}{2}(3\, n_o^{(k)2} - \frac{1}{4})\xi_{3,\alpha} + \frac{3h^2}{2}(5\, n_o^{(k)2} - \frac{1}{4})\phi_\alpha \Big] \tag{16a}$$

$$\frac{1}{h} R_\alpha^{(k)} - \frac{n^{(k)2}h}{40} (T_\alpha^{(k-1)} - T_\alpha^{(k)}) = \frac{7h^2}{120} n^{(k)3} \bar{C}_\alpha^{(k)} \left[\Psi_{3,\alpha} + 3\xi_\alpha + S_{3,\alpha} (-1)^k \frac{2}{n^{(k)}h} \right.$$

$$\left. + 3h\, n_o^{(k)} (\xi_{3,\alpha} + 5\phi_\alpha) \right] \tag{16b}$$

$$Q_\alpha^{(k)} - \frac{14 J_\alpha^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{12} (T_\alpha^{(k-1)} + T_\alpha^{(k)}) = -\frac{3h^2}{40} n^{(k)3} \bar{C}_\alpha^{(k)} (\xi_{3,\alpha} + 5\phi_\alpha) \tag{16c}$$

$$-\frac{1}{\bar{C}_\alpha^{(k)}} \left[\frac{1}{12} Q_\alpha^{(k)} - \frac{5 J_\alpha^{(k)}}{3(n^{(k)}h)^2} + \frac{3 R_\alpha^{(k)}}{7 n^{(k)}h} \right] - \frac{1}{\bar{C}_\alpha^{(k+1)}} \left[\frac{1}{12} Q_\alpha^{(k+1)} \right.$$

$$\left. - \frac{5 J_\alpha^{(k+1)}}{3 (n^{(k+1)}h)^2} - \frac{3 R_\alpha^{(k+1)}}{7 n^{(k+1)}h} \right] \tag{16d}$$

$$= \frac{h}{126} \left[\frac{-n^{(k)}}{\bar{C}_\alpha^{(k)}} T_\alpha^{(k-1)} + 8 \left(\frac{n^{(k)}}{\bar{C}_\alpha^{(k)}} + \frac{n^{(k+1)}}{\bar{C}_\alpha^{(k+1)}} \right) T_\alpha^{(k)} - \frac{n^{(k+1)}}{\bar{C}_\alpha^{(k+1)}} T_\alpha^{(k+1)} \right]$$

- For Normal Stresses

$$Q_3^{(k)} - \frac{8 J_3^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{30} (T_3^{(k-1)} + T_3^{(k)}) = \frac{2h}{5} n^{(k)} C_{33}^{(k)} \left[\Psi_3 + S_3 (-1)^k \frac{2}{n^{(k)}h} + 3h\, n_o^{(k)} \xi_3 \right]$$

$$+ \frac{2h}{5} n^{(k)} \left[\bar{U} + h\, n_o^{(k)} \bar{\Psi} + \frac{h^2}{2} (3 n_o^{(k)2} - \frac{1}{4}) \bar{\xi} + \frac{h^3}{2} \left(5 n_o^{(k)3} - \frac{3}{4} n_o^{(k)} \right) \bar{\phi} \right] \tag{17a}$$

$$\frac{1}{h} R_3^{(k)} - \frac{32 I_3^{(k)}}{5 n^{(k)2} h^3} + \frac{n^{(k)2}h}{140} (T_3^{(k-1)} - T_3^{(k)}) = \frac{11}{350} h^2 n^{(k)3} C_{33}^{(k)} \xi_3$$

$$+ \frac{11}{1050} h^2 n^{(k)3} \left[\bar{\Psi} + (-1)^k \frac{2}{n^{(k)}h} \bar{S} + 3h\, n_o^{(k)} \bar{\xi} + \frac{3h^2}{2} (5 n_o^{(k)2} - \frac{1}{4}) \bar{\phi} \right] \tag{17b}$$

$$Q_3^{(k)} - \frac{14 J_3^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{12} (T_3^{(k-1)} + T_3^{(k)}) = -\frac{3h^3}{40} n^{(k)3} [\bar{\xi} + 5h\, n_o^{(k)} \bar{\phi}] \tag{17c}$$

$$\frac{1}{h} R_3^{(k)} - \frac{15 I_3^{(k)}}{2 n^{(k)2} h^3} + \frac{n^{(k)2}h}{96} (T_3^{(k-1)} - T_3^{(k)}) = \frac{-11 h^4}{2688} n^{(k)5} \bar{\phi} \tag{17d}$$

$$\frac{-11}{12}\left[\frac{Q_3^{(k)}}{C_{33}^{(k)}}+\frac{Q_3^{(k+1)}}{C_{33}^{(k+1)}}\right]+\frac{15}{2h}\left[\frac{R_3^{(k)}}{n^{(k)}C_{33}^{(k)}}-\frac{R_3^{(k+1)}}{n^{(k+1)}C_{33}^{(k+1)}}\right]+\frac{5}{3h^2}\left[\frac{J_3^{(k)}}{n^{(k)2}C_{33}^{(k)}}+\frac{J_3^{(k+1)}}{n^{(k+1)2}C_{33}^{(k+1)}}\right]$$

$$\frac{-70}{11h^3}\left[\frac{I_3^{(k)}}{n^{(k)3}C_{33}^{(k)}}-\frac{I_3^{(k+1)}}{n^{(k+1)3}C_{33}^{(k+1)}}\right]=\frac{h}{18}\left[\frac{n^{(k)}}{C_{33}^{(k)}}T_3^{(k-1)}+10\left(\frac{n^{(k)}}{C_{33}^{(k)}}+\frac{n^{(k+1)}}{C_{33}^{(k+1)}}\right)T_3^{(k)}+\frac{n^{(k+1)}}{C_{33}^{(k+1)}}T_3^{(k+1)}\right] \tag{17e}$$

where in (16a,b,c) and (17a,b,c,d) k ranges from 1 to N while in (16d) and (17e) k ranges from 1 to $(N-1)$. Also, no summation on α is implied in (16) and

$$\tilde{C}_\alpha^{(k)} \equiv \delta_{\alpha 1} C_{55}^{(k)} + \delta_{\alpha 2} C_{44}^{(k)} \;\; ; \;\; n_o^{(k)} \equiv x_{30}^{(k)}/h \tag{18}$$

$$\begin{bmatrix} \tilde{U} \\ \tilde{\Psi} \\ \tilde{S} \\ \tilde{\xi} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} U_{1,1} & U_{2,2} \\ \Psi_{1,1} & \Psi_{2,2} \\ S_{1,1} & S_{2,2} \\ \xi_{1,1} & \xi_{2,2} \\ \phi_{1,1} & \phi_{2,2} \end{bmatrix} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix}^{(k)} \tag{19}$$

By solving (16) and (17), $Q_i^{(k)}$, $R_i^{(k)}$, $J_i^{(k)}$, $I_3^{(k)}$ and $T_i^{(k)}$ are obtained in terms of U_i, Ψ_i, S_i, ξ_i and ϕ_α and their derivatives. As a result, the quantities N_{3i}, M_{3i}, K_{3i}, L_{3i} of Eq. (15c) can be determined as functions of these displacement variables. Such expressions will automatically include the appropriate shear correction factors by virtue of the Reissner mixed variational principle.

The equilibrium equations (14) are supplemented with the following suitable boundary conditions:

$$\text{specify} \quad U_i \quad \text{or} \quad N_{\alpha i}\, \nu_\alpha \;, \tag{20a}$$

$$\text{specify} \quad \Psi_i \quad \text{or} \quad M_{\alpha i}\, \nu_\alpha \;, \tag{20b}$$

$$\text{specify} \quad S_i \quad \text{or} \quad Z_{\alpha i}\, \nu_\alpha \;, \tag{20c}$$

$$\text{specify} \quad \xi_i \quad \text{or} \quad L_{\alpha i}\, \nu_\alpha \;, \tag{20d}$$

$$\text{specify} \quad \phi_\alpha \quad \text{or} \quad P_{\beta\alpha}\, \nu_\beta \;. \tag{20e}$$

The remaining constitutive equations for $N_{\alpha\beta}$, $M_{\alpha\beta}$, $Z_{\alpha\beta}$, $L_{\alpha\beta}$ and $P_{\alpha\beta}$ are obtained by substituting

(3a), (4), (8) and (10b) into (15a) to yield:

$$\begin{bmatrix} \frac{1}{h}\,\underset{\sim}{N} \\ \frac{1}{h^2}\,\underset{\sim}{M} \\ \frac{1}{h}\,\underset{\sim}{Z} \\ \frac{1}{h^3}\,\underset{\sim}{L} \\ \frac{1}{h^4}\,\underset{\sim}{P} \end{bmatrix} = \begin{bmatrix} [N_u] & [N_\Psi] & 0 & [N_\xi] & [N_\phi] \\ & [M_\Psi] & [M_s] & [M_\xi] & [M_\phi] \\ & & \frac{1}{3}[N_u] & [Z_\xi] & [Z_\phi] \\ \text{symmetric} & & & [L_\xi] & [L_\phi] \\ & & & & [P_\phi] \end{bmatrix} \begin{bmatrix} \underset{\sim}{U} \\ h\,\underset{\sim}{\Psi} \\ \underset{\sim}{S} \\ h^2\,\underset{\sim}{\xi} \\ h^3\,\underset{\sim}{\phi} \end{bmatrix} + \sum_{k=1}^{N} [C]^{(k)} \begin{bmatrix} \underset{\sim}{V}^N \\ h\underset{\sim}{V}^M \\ \underset{\sim}{V}^Z \\ h^2\underset{\sim}{V}^L \\ h^3\underset{\sim}{V}^P \end{bmatrix} \begin{bmatrix} Q_3 \\ \frac{1}{h}\,R_3 \\ \frac{1}{h^2}\,J_3 \\ \frac{1}{h^3}\,I_3 \end{bmatrix}^{(k)} \qquad (21)$$

where $\underset{\sim}{N} = [N_{11}\ N_{22}\ N_{12}]^T$, $\underset{\sim}{U} = [U_{1,1}\ U_{2,2}U_{1,2} + U_{2,1}]^T$ with analogous expressions for $\underset{\sim}{M}$, $\underset{\sim}{\Psi}$, ..., $\underset{\sim}{P}$, $\underset{\sim}{\phi}$ · $[N_u]$,..., $[P_\phi]$ are 3×3 matrices, $[C]^{(k)}$ is a 15×5 matrix an $\underset{\sim}{V}^N$, ..., $\underset{\sim}{V}^P$ are 1×4 vectors, which are given in the Appendix.

5. CYLINDRICAL BENDING OF LAMINATED PLATES

In order to test the accuracy of the present theory, cylindrical bending of composite plates under sinusoidal loading is considered. The plate is simply supported at the ends $x_1 = 0$ and l and is infinitely long in the x_2-direction. The prescribed boundary conditions on the top and bottom surfaces of the plate are:

$$T_1^+ = 0\ ,\ T_3^+ = q \sin \frac{\pi x_1}{l} \qquad \text{on } x_3 = \frac{h}{2} \tag{22a}$$

$$T_1^- = T_3^- = 0 \qquad \text{on } x_3 = -\frac{h}{2}\ . \tag{22b}$$

The boundary conditions for the simply supported ends are, from (20):

$$U_3 = \Psi_3 = S_3 = \xi_3 = 0 \qquad \text{at } x_1 = 0\ ,\ l \tag{23a}$$

$$N_{11} = M_{11} = Z_{11} = L_{11} = P_{11} = 0 \qquad \text{at } x_1 = 0\ ,\ l\ . \tag{23b}$$

Using surface boundary conditions (22), the equilibrium equations (14) for cylindrical bending reduce to:

$$N_{11,1} = 0 \tag{24a}$$

$$N_{13,1} + q \sin \frac{\pi x_1}{l} = 0 \tag{24b}$$

$$M_{11,1} - N_{31} = 0 \tag{24c}$$

$$M_{13,1} - N_{33} + \frac{h}{2} q \sin \frac{\pi x_1}{l} = 0 \tag{24d}$$

$$Z_{11,1} - K_{31} = 0 \tag{24e}$$

$$Z_{13,1} - K_{33} - q \sin \frac{\pi x_1}{l} = 0 \tag{24f}$$

$$L_{11,1} - 3M_{31} = 0 \tag{24g}$$

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$$L_{13,1} - 3M_{33} + \frac{h^2}{4} q \sin \frac{\pi x_1}{l} = 0 \tag{24h}$$

$$P_{11,1} - 5L_{31} - \frac{h^2}{4} N_{31} = 0 \tag{24i}$$

From the boundary condition $N_{11} = 0$ at $x_1 = 0, l$, Eq. (24a) implies that

$$N_{11} = 0 \ . \tag{25}$$

Next Eqs. (15a,c) are expressed in terms of the displacement variables U_1, ..., ξ_3. To this end, the constitutive equations (16) and (17), for the cylindrical bending analysis, can be rewritten in the following vector form:

$$\underset{\sim}{Q}_1 - \frac{1}{h^2} \underset{\sim}{\bar{J}}_1 + h [A_1] \underset{\sim}{T}_1 = \underset{\sim}{\lambda}_1 \tag{26a}$$

$$\frac{1}{h} \underset{\sim}{R}_1 + h [B_1] \underset{\sim}{T}_1 = \underset{\sim}{\lambda}_2 \tag{26b}$$

$$\underset{\sim}{Q}_1 - \frac{7}{4} \frac{1}{h^2} \underset{\sim}{\bar{J}}_1 + \frac{5}{2} h [A_1] \underset{\sim}{T}_1 = \underset{\sim}{\lambda}_3 \tag{26c}$$

$$[TQ_1] \underset{\sim}{Q}_1 + \frac{1}{h} [TR_1] \underset{\sim}{R}_1 - \frac{5}{2} \frac{1}{h^2} [TQ_1] \underset{\sim}{\bar{J}}_1 = h [C_1] \underset{\sim}{T}_1 \tag{26d}$$

and

$$\underset{\sim}{Q}_3 - \frac{1}{h^2} \underset{\sim}{\bar{J}}_3 + h [A_1] \underset{\sim}{T}_3 = \underset{\sim}{\kappa}_1 \tag{27a}$$

$$\frac{1}{h} \underset{\sim}{R}_3 - \frac{1}{h^3} \underset{\sim}{\bar{I}}_3 - \frac{2}{7} h [B_1] \underset{\sim}{T}_3 = \underset{\sim}{\kappa}_2 \tag{27b}$$

$$\underset{\sim}{Q}_3 - \frac{7}{4} \frac{1}{h^2} \underset{\sim}{\bar{J}}_3 + \frac{5}{2} h [A_1] \underset{\sim}{T}_3 = \underset{\sim}{\kappa}_3 \tag{27c}$$

$$\frac{1}{h} \underset{\sim}{R}_3 - \frac{75}{64} \frac{1}{h^3} \underset{\sim}{\bar{I}}_3 - \frac{5}{12} h [B_1] \underset{\sim}{T}_3 = \underset{\sim}{\kappa}_4 \tag{27d}$$

$$[TQ_3] \underset{\sim}{Q}_3 + \frac{1}{h} [TR_3] \underset{\sim}{R}_3 - \frac{5}{2} \frac{1}{h^2} [TQ_3] \underset{\sim}{\bar{J}}_3 - \frac{35}{24} \frac{1}{h^3} [TR_3] \underset{\sim}{\bar{I}}_3 = h [C_3] \underset{\sim}{T}_3 \tag{27e}$$

$$\text{where} \quad \underset{\sim}{\bar{J}}_i \equiv \frac{8 \underset{\sim}{J}_i}{n^{(k)2}} \quad i = 1,3 \quad \text{and} \quad \underset{\sim}{\bar{I}}_3 \equiv \frac{32}{5\, n^{(k)2}} \underset{\sim}{I}_3 \ . \tag{28}$$

The matrices $[A_1]$, ..., $[C_3]$ and vectors $\underset{\sim}{\lambda}_1$, ..., $\underset{\sim}{\kappa}_4$ are given in the Appendix. The vector equations (26a,b,c) and (27a,b,c,d) have N-components, while the vector equations (26d) and (27e) have $(N-1)$ components. Matrices $[A_1]$, ..., $[C_3]$ depend on the volume fractions $n^{(k)}$ and elastic constants $C_{55}^{(k)}$, $C_{13}^{(k)}$ and $C_{33}^{(k)}$, while the vectors $\underset{\sim}{\lambda}_1$, ..., $\underset{\sim}{\kappa}_4$ contain the displacement variables U_1, ..., ξ_3.

Eqs. (26) are easily solved by substituting $\underset{\sim}{Q}_1$, $\frac{1}{h}\underset{\sim}{R}_1$ and $\frac{1}{h^2}\underset{\sim}{\bar{J}}_1$ in terms of $\underset{\sim}{T}_1$ from (26a,b,c) into (26d). This yields a new equation involving $\underset{\sim}{T}_1$ only, which can thus be solved for $\underset{\sim}{T}_1$. Then by back substitution expressions for $\underset{\sim}{Q}_1$, $\frac{1}{h}\underset{\sim}{R}_1$ and $\frac{1}{h^2}\underset{\sim}{\bar{J}}_1$ in terms of $\underset{\sim}{\lambda}_1$, $\underset{\sim}{\lambda}_2$ and $\underset{\sim}{\lambda}_3$ are obtained. Proceeding in a similar manner with (27a,b,c,d) $\underset{\sim}{Q}_3$, $\frac{1}{h}\underset{\sim}{R}_3$, $\frac{1}{h^2}\underset{\sim}{\bar{J}}_3$ and $\frac{1}{h^3}\underset{\sim}{\bar{I}}_3$ in terms of $\underset{\sim}{\kappa}_1$, $\underset{\sim}{\kappa}_2$, $\underset{\sim}{\kappa}_3$ and $\underset{\sim}{\kappa}_4$ are determined. These expressions are:

$$\begin{bmatrix} \underset{\sim}{Q}_1 \\ \frac{1}{h^2}\underset{\sim}{\bar{J}}_1 \end{bmatrix} = \begin{bmatrix} (\frac{7}{3}[I]-[AQ_1]) & -(\frac{4}{3}[I]-2[AQ_1]) \\ (\frac{4}{3}[I]-2[AQ_1]) & -(\frac{4}{3}[I]-4[AQ_1]) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\lambda}_1 \\ \underset{\sim}{\lambda}_3 \end{bmatrix} + \begin{bmatrix} [AR_1] \\ 2[AR_1] \end{bmatrix} \underset{\sim}{\lambda}_2 \tag{29a}$$

$$\frac{1}{h}\underset{\sim}{R}_1 = [BQ_1](\underset{\sim}{\lambda}_1 - 2\underset{\sim}{\lambda}_3) + ([I]-[BR_1])\underset{\sim}{\lambda}_2 \tag{29b}$$

and

$$\begin{bmatrix} \underset{\sim}{Q}_3 \\ \frac{1}{h^2}\underset{\sim}{\bar{J}}_3 \end{bmatrix} = \begin{bmatrix} (\frac{7}{3}[I]-[AQ_3]) & -(\frac{4}{3}[I]-2[AQ_3]) \\ (\frac{4}{3}[I]-2[AQ_3]) & -(\frac{4}{3}[I]-4[AQ_3]) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\kappa}_1 \\ \underset{\sim}{\kappa}_3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} [AR_3] \\ 2[AR_3] \end{bmatrix} (8\underset{\sim}{\kappa}_4 - 5\underset{\sim}{\kappa}_2) \tag{30a}$$

$$\begin{bmatrix} \frac{1}{h}\underset{\sim}{R}_3 \\ \frac{1}{h^3}\underset{\sim}{\bar{I}}_3 \end{bmatrix} = \begin{bmatrix} (\frac{75}{11}[I]+\frac{50}{63}[BR_3]) & -(\frac{64}{11}[I]+\frac{80}{63}[BR_3]) \\ (\frac{64}{11}[I]+\frac{80}{63}[BR_3]) & -(\frac{64}{11}[I]+\frac{128}{63}[BR_3]) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\kappa}_2 \\ \underset{\sim}{\kappa}_4 \end{bmatrix} - \frac{1}{21}\begin{bmatrix} 10[BQ_3] \\ 16[BQ_3] \end{bmatrix} (2\underset{\sim}{\kappa}_3 - \underset{\sim}{\kappa}_1) \tag{30b}$$

where $[I]$ is the $N \times N$ identity matrix and

$$\begin{bmatrix} [AQ_i] & [AR_i] \\ [BQ_i] & [BR_i] \end{bmatrix} = \begin{bmatrix} [A_1] \\ [B_1] \end{bmatrix} [TV_i] \begin{bmatrix} [TQ_i] & [TR_i] \end{bmatrix} \quad i = 1,3 \tag{31a}$$

with

with

$$[TV_1] = (4[TQ_1][A_1] + [TR_1][B_1] + [C_1])^{-1} \tag{31b}$$

$$[TV_3] = (4[TQ_3][A_1] - \frac{40}{63}[TR_3][B_1] + [C_3])^{-1} \tag{31c}$$

$[AQ_i]$, ..., $[BR_i]$ are $N \times N$ matrices, while $[TV_i]$ are $(N-1) \times (N-1)$ matrices. By inserting (29) and (30) into (15c) and (21) the appropriate constitutive relations for the cylindrical bending problem in terms of the displacement variables U_1, ..., ξ_3 and their derivatives with respect to x_1 are obtained.

The form of the dependence on the displacement variables U_1, ..., ξ_3 of the constitutive equations thus obtained and the nature of the applied load suggest the following expressions for the displacements:

$$\begin{bmatrix} U_1 \\ \Psi_1 \\ S_1 \\ \xi_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} h\hat{U}_1 \\ \hat{\Psi}_1 \\ h\hat{S}_1 \\ \hat{\xi}_1/h \\ \hat{\phi}_1/h^2 \end{bmatrix} \cos \pi \frac{x_1}{l} \quad \text{and} \quad \begin{bmatrix} U_3 \\ \Psi_3 \\ S_3 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} h\hat{U}_3 \\ \hat{\Psi}_3 \\ h\hat{S}_3 \\ \hat{\xi}_3/h \end{bmatrix} \sin \pi \frac{x_1}{l} \tag{32}$$

where the "^" quantities are nondimensional by definition. It is easily proven that the boundary conditions (23) are satisfied when (32) are substituted therein.

Finally, inserting (32) into the constitutive equations obtained in the manner described above and these in turn into the equilibrium equations (24) and (25) yields a system of nine algebraic equations with the nine nondimensional quantities $\hat{U}_1$, ..., $\hat{\xi}_3$ as unknowns. This system is conveniently written in matrix form as

$$[B]\,\underset{\sim}{U} = \underset{\sim}{F} \tag{33}$$

where $\underset{\sim}{U} = [\hat{U}_1\ \hat{\Psi}_1\ \hat{S}_1\ \hat{\xi}_1\ \hat{\phi}_1\ \hat{U}_3\ \hat{\Psi}_3\ \hat{S}_3\ \hat{\xi}_3]^T$ (34a)

$$\underset{\sim}{F} = [0, q, 0, \frac{1}{2}q, 0, -q, 0, \frac{1}{4}q, 0]^T \tag{34b}$$

and $[B]$ is a 9×9 matrix.

6. NUMERICAL RESULTS

In order to assess the accuracy of the present theory the problem of the cylindrical bending of an infinitely long strip under sinusoidal loading is examined. The exact elasticity solution has been given by Pagano [1], where a three layer cross-ply laminate was considered, the 0° layers being at the outer surfaces of the laminate. The elastic properties are:

$$\text{for the } 0^\circ \text{ layers} \qquad \frac{\tilde{C}_{11}}{E_T} = 25.062657 \ , \quad \frac{C_{13}}{E_T} = 0.335570$$

$$\frac{C_{33}}{E_T} = 1.071141 \ , \quad \frac{C_{55}}{E_T} = 0.5; \tag{35a}$$

$$\text{and for the } 90^\circ \text{ layers} \qquad \frac{\tilde{C}_{11}}{E_T} = 1.002506 \ , \quad \frac{C_{13}}{E_T} = 0.271141$$

$$\frac{C_{33}}{E_T} = 1.071141 \ , \quad \frac{C_{55}}{E_T} = 0.2$$

where E_T is a reference modulus.

We follow Pagano's [1] nondimensionalization and write the displacements and stresses in the form

$$\bar{u}_1^{(k)} = \left(\frac{E_T}{q}\right) \frac{u_1^{(k)}(0,x_3)}{h} \ , \ \bar{u}_3^{(k)} = \left(\frac{E_T}{q}\right) \frac{100h^3}{l^4} u_3^{(k)}\left(\frac{l}{2}, 0\right) \tag{36}$$

$$\bar{\sigma}_{11}^{(k)} = \frac{1}{q}\sigma_{11}^{(k)}\left(\frac{l}{2}, x_3\right)$$

$$\text{Also} \qquad \bar{x}_3 = \frac{x_3}{h} \ , \ S = \frac{l}{h} \tag{37}$$

In the various curves the solid line represents the exact solution while the results of the present theory are shown by a broken line. Also shown, for comparison purposes, are the results given by the first order zig-zag model [11] and Lo, Christensen and Wu's high-order theory (LCW) [7], which are

represented by a dashed-dotted line and dotted solid line, respectively. Symmetric 3, 5 and 9-ply laminates and asymmetric 4 and 8-ply laminates were examined, to test the present theory.

For a symmetric 3-ply laminate (0/90/0) with layers of equal thickness, Table 1 shows the values of the central deflection $\bar{u}_3$ obtained from the different theories for a span-to-thickness ratio S of 4 and 6. As observed the present high-order theory correctly predicts the central deflection $\bar{u}_3$ to the first two decimal digits, while the first order zig-zag model gives a better result than LCW. The variation of the in-plane displacement $\bar{u}_1$ across the plate thickness is compared in Fig. 2a for $S = 4$, where it is seen that the curves for the present theory and the exact solution are almost identical. This improvement is also reflected in the variation of the in-plane stress $\bar{\sigma}_{11}$ across the plate thickness, as shown in Fig. 2b. Very close agreement is found between Pagano's exact solution and the present theory, which has improved upon Lo, Christensen and Wu's high-order theory, especially at and in the neighborhood of the interfaces.

The present theory was next tested for a symmetric 5-ply laminate (0/90/0/90/0) with layers of equal thickness. The central deflection $\bar{u}_3$ for span-to-thickness ratio S of 4 and 6, is shown in Table 1 where close agreement with the exact solution is observed. The variations across the plate thickness of in-plane variables $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are compared in Figs. 3 and 4. The curves for the present high-order theory and the exact solution are again almost identical. In particular, it is seen that the present theory has considerably improved upon Lo, Christensen and Wu's model in the interior layers of the plate.

To further assess the accuracy of the present high-order theory the more difficult case of a symmetric 9-layer cross-ply laminate (0/90/0/90/0/90/0/90/0) was considered. The 0° layers have equal thickness $h/10$ while the 90° layers have equal thickness $h/8$. The results for the central deflection $\bar{u}_3$ are given in Table 1 for $S = 4$ and 6 where again close agreement with the exact solution is observed. The variations across the plate thickness of the in-plane displacement $\bar{u}_1$ and normal stress $\bar{\sigma}_{11}$ are shown in Figs. 5 and 6, for $S = 4$ and 6 respectively. There the discrepancies between the first order zig-zag theory and the exact solution are more pronounced than in the 3- and 5-layer cases, as

expected. However, the results of the present theory are still very good when compared to the exact solution.

Finally, asymmetric 4 and 8 cross-ply laminates, with layers of equal thickness, were examined. The present theory predicts accurately the central deflection $\bar{u}_3$. These results are given in Table 2 for span-to-thickness ratio S of 4 and 6. The variation across the plate thickness of the in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$ are shown in Figs. 7, 8 and 9 for $S = 4$ and 6. From the curves for $\bar{u}_1^{(k)}$, it is seen that the first-order zig-zag theory deviates significantly from the exact solution at the bottom layer of the plate. On the other hand, the discrepancies between LCW and the exact solution, for both $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are more pronounced in the interior layers of the plate, while the present high-order theory is still in very good agreement with the exact solution.

7. CONCLUSION

A high-order laminated plate theory, which accurately predicts in-plane responses of symmetric and asymmetric laminates, was developed with the help of Reissner's new mixed variational principle [10]. The improvement was achieved by including a zig-zag shaped C° function in the in-plane displacement variations across the plate thickness, as proposed by Murakami [11], while the non-linear variation is accounted for by using Legendre Polynomials. The accuracy of the theory was examined for the case of cylindrical bending of an infinitely long strip and compared with the exact elasticity solution given by Pagano [1]. The results for the central deflection and in-plane displacements and normal stresses for several symmetric and asymmetric cross-ply laminates indicate that the theory very accurately predicts these in-plane responses even for small span-to-thickness ratios. In all the cases considered, the proposed theory gave better in-plane responses than the Lo, Christensen and Wu high-order theory, especially in the interior layers of the plate. It was also observed that for symmetric laminates, the first order zig-zag model [11] predicts more accurately the central deflection than the Lo, Christensen and Wu high-order theory.

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APPENDIX

- Matrices $[N_u]$, ..., $[P_\phi]$ in Eq. (21):

$$[N_u] = \begin{bmatrix} D_2 & D_2' & 0 \\ D_2' & D_2'' & 0 \\ 0 & 0 & D_2''' \end{bmatrix}, \ [N_\Psi] = \begin{bmatrix} C_1 & C_1' & 0 \\ C_1' & C_1'' & 0 \\ 0 & 0 & C_1''' \end{bmatrix},$$

$$[N_\xi] = \begin{bmatrix} C_8 & C_8' & 0 \\ C_8' & C_8'' & 0 \\ 0 & 0 & C_8''' \end{bmatrix}, \ [N_\phi] = \begin{bmatrix} C_9 & C_9' & 0 \\ C_9' & C_9'' & 0 \\ 0 & 0 & C_9''' \end{bmatrix}$$

$$[M_\Psi] = \begin{bmatrix} C_2 & C_2' & 0 \\ C_2' & C_2'' & 0 \\ 0 & 0 & C_2''' \end{bmatrix}, \ [M_s] = \begin{bmatrix} C_3 & C_3' & 0 \\ C_3' & C_3'' & 0 \\ 0 & 0 & C_3''' \end{bmatrix}, \tag{A1}$$

$$[M_\xi] = \begin{bmatrix} C_7 & C_7' & 0 \\ C_7' & C_7'' & 0 \\ 0 & 0 & C_7''' \end{bmatrix}, \ [M_\phi] = \begin{bmatrix} C_6 & C_6' & 0 \\ C_6' & C_6'' & 0 \\ 0 & 0 & C_6''' \end{bmatrix}$$

$$[Z_\xi] = \begin{bmatrix} D_3 & D_3' & 0 \\ D_3' & D_3'' & 0 \\ 0 & 0 & D_3''' \end{bmatrix}, \ [Z_\phi] = \begin{bmatrix} D_1 & D_1' & 0 \\ D_1' & D_1'' & 0 \\ 0 & 0 & D_1''' \end{bmatrix},$$

$$[L_\xi] = \begin{bmatrix} D_5 & D_5' & 0 \\ D_5' & D_5'' & 0 \\ 0 & 0 & D_5''' \end{bmatrix}, \ [L_\phi] = \begin{bmatrix} F_3 & F_3' & 0 \\ F_3' & F_3'' & 0 \\ 0 & 0 & F_3''' \end{bmatrix}$$

$$[P_\phi] = \begin{bmatrix} F_4 & F_4' & 0 \\ F_4' & F_4'' & 0 \\ 0 & 0 & F_4''' \end{bmatrix}$$

where

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} n_o^{(k)}\, n^{(k)} \\ n^{(k)3}/12 + n_o^{(k)2}\, n^{(k)} \\ (-1)^k\, n^{(k)2}/6 \end{bmatrix}, \ \begin{bmatrix} D_2 \\ \\ D_3 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} n^{(k)} \\ \\ (-1)^k\, n_o^{(k)}\, n^{(k)2}/2 \end{bmatrix}$$

and

$$C_6 = \frac{5}{2} C_5 - \frac{3}{8} C_2 \,,\; C_7 = \frac{3}{2} C_4 - \frac{1}{8} C_1 \,,\; C_8 = \frac{3}{2} C_2 - \frac{1}{8} D_2 \,,\; C_9 = \frac{5}{2} C_4 - \frac{3}{8} C_1$$

$$D_1 = D_4 - \frac{3}{8} C_3 \,,\; D_5 = \frac{9}{4} C_5 - \frac{3}{8} C_2 + \frac{1}{64} D_2 \tag{A3}$$

$$F_3 = \frac{15}{4} F_1 - \frac{7}{8} C_4 + \frac{3}{64} C_1 \,,\; F_4 = \frac{25}{4} F_2 - \frac{15}{8} C_5 + \frac{9}{64} C_2$$

where

$$\begin{bmatrix} C_4 \\ C_5 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} \frac{1}{4} n_o^{(k)} n^{(k)3} + n_o^{(k)3} n^{(k)} \\ \frac{1}{80} n^{(k)5} + \frac{1}{2} n_o^{(k)2} n^{(k)3} + n_o^{(k)4} n^{(k)} \end{bmatrix} \,,\quad D_4 = \sum_k \tilde{C}_{11}^{(k)} (-1)^k \left(\frac{n^{(k)4}}{16} + \frac{5}{4} n_o^{(k)2} n^{(k)2} \right)$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \sum_k \tilde{C}_{11}^{(k)} \begin{bmatrix} \frac{n_o^{(k)} n^{(k)5}}{16} + \frac{5}{6} n_o^{(k)3} n^{(k)3} + n_o^{(k)5} n^{(k)} \\ \frac{n^{(k)7}}{448} + \frac{3}{16} n_o^{(k)2} n^{(k)5} + \frac{5}{4} n_o^{(k)4} n^{(k)3} + n_o^{(k)6} n^{(k)} \end{bmatrix} \tag{A4}$$

The ( )′, ( )″ and ( )‴ quantities can be obtained from (A1,2,3,4) by replacing therein $\tilde{C}_{11}^{(k)}$ by $\tilde{C}_{12}^{(k)}$, $\tilde{C}_{22}^{(k)}$ and $\tilde{C}_{66}^{(k)}$ respectively, where k ranges from 1 to N.

- Matrix $[C]^{(k)}$ and Vectors $\underset{\sim}{V}^N$, ..., $\underset{\sim}{V}^P$ in Eq. (21):

$$\underset{(15\times 4)}{[C]^{(k)}} = \begin{bmatrix} \underset{\sim}{c} & & & & \\ & \underset{\sim}{c} & 0 & & \\ & & \underset{\sim}{c} & & \\ & 0 & & \underset{\sim}{c} & \\ & & & & \underset{\sim}{c} \end{bmatrix}^{(k)} \quad \text{where} \quad \underset{\sim}{c}^{(k)} = [C_{13}/C_{33} \quad C_{23}/C_{33} \quad 0]^T \tag{A5}$$

$$\underset{\sim}{V}^N = [1,0,0,0] \;;\; \underset{\sim}{V}^M = [n_o^{(k)}, 1,0,0] \;;\; \underset{\sim}{V}^Z = [0,(-1)^k \frac{2}{n^{(k)}}, 0, 0]$$

$$\underset{\sim}{V}^L = [\frac{1}{2}(3n_o^{(k)2} - \frac{1}{4}), 3n_o^{(k)}, \frac{3}{2}, 0] \;;\; \underset{\sim}{V}^P = [\frac{1}{2}(5n_o^{(k)3} - \frac{3n_o^{(k)}}{4}), \frac{3}{2}(5n_o^{(k)2} - \frac{1}{4}), \frac{15}{2} n_o^{(k)}, \frac{5}{2}] \tag{A6}$$

- Matrices $[A_1]$, ..., $[C_3]$ in Eqs. (26) and (27):

$$\underset{(N\times N-1)}{[A_1]} = \frac{1}{30} \begin{bmatrix} & 0 \\ n^{(k)} & n^{(k)} \\ 0 & \end{bmatrix} \,,\quad \underset{(N\times N-1)}{[B_1]} = \frac{1}{40} \begin{bmatrix} & 0 \\ -n^{(k)2} & n^{(k)2} \\ 0 & \end{bmatrix}$$

$$
,\quad \underset{(N-1\times N-1)}{[C_1]} = \frac{1}{126}\begin{bmatrix} & & 0 \\ \dfrac{-n^{(k)}}{C_{55}^{(k)}} & 8\left(\dfrac{n^{(k)}}{C_{55}^{(k)}} + \dfrac{n^{(k+1)}}{C_{55}^{(k+1)}}\right) & \dfrac{-n^{(k+1)}}{C_{55}^{(k+1)}} \\ & 0 & \end{bmatrix}
$$

$$
\underset{(N-1)\times N}{[TQ_1]} = \frac{1}{12}\begin{bmatrix} & 0 \\ \dfrac{-1}{C_{55}^{(k)}} & \dfrac{-1}{C_{55}^{(k+1)}} \\ 0 & \end{bmatrix} ,\quad \underset{(N-1\times N)}{[TR_1]} = \frac{3}{7}\begin{bmatrix} & 0 \\ \dfrac{-1}{n^{(k)}C_{55}^{(k)}} & \dfrac{1}{n^{(k+1)}C_{55}^{(k+1)}} \\ 0 & \end{bmatrix}
$$

$$
,\quad \underset{(N-1)\times N}{[TQ_3]} = \frac{11}{12}\begin{bmatrix} & 0 \\ \dfrac{1}{C_{33}^{(k)}} & \dfrac{1}{C_{33}^{(k+1)}} \\ 0 & \end{bmatrix}
$$

$$
\underset{(N-1\times N)}{[TR_3]} = \frac{-15}{2}\begin{bmatrix} & 0 \\ \dfrac{1}{n^{(k)}C_{33}^{(k)}} & \dfrac{1}{n^{(k+1)}C_{33}^{(k+1)}} \\ 0 & \end{bmatrix} \quad \underset{(N-1\times N-1)}{[C_3]} = -\frac{1}{18}\begin{bmatrix} & & 0 \\ \dfrac{n^{(k)}}{C_{33}^{(k)}} & 10\left(\dfrac{n^{(k)}}{C_{33}^{(k)}} + \dfrac{n^{(k+1)}}{C_{33}^{(k+1)}}\right) & \dfrac{n^{(k+1)}}{C_{55}^{(k+1)}} \\ 0 & & \end{bmatrix}
$$

- Vectors $\underline{\lambda}_1, \ldots, \underline{\kappa}_4$ in Eqs. (26) and (27):

$$\underline{\lambda}_1 = h\,(U_{3,1}+\Psi_1)\,\underline{a}_1 + S_1\,\underline{b}_1 + h^2(\Psi_{3,1}+3\xi_1)\underline{c}_1 + h^3\xi_{3,1}\,\underline{d}_1 + h^3\phi_1\,\underline{e}_1$$

$$\underline{\lambda}_2 = h^2(\Psi_{3,1}+3\xi_1)\underline{f}_1 + hS_{3,1}\,\underline{g}_1 + h^3(\xi_{3,1}+5\phi_1)\underline{p}_1$$

$$\underline{\lambda}_3 = \frac{-9}{7}\,h^3\,(\xi_{3,1}+5\phi_1)\underline{f}_1 \qquad \text{(A8)}$$

$$\underline{\kappa}_1 = hU_{1,1}\,\underline{a}_2 + h\Psi_3\,\underline{a}_3 + S_3\,\underline{b}_3 + h^2\xi_3\underline{c}_2 + h^2\Psi_{1,1}\,\underline{c}_3 + h^3\xi_{1,1}\,\underline{d}_3 + h^4\phi_{1,1}\,\underline{e}_3$$

$$\underline{\kappa}_2 = h^2\xi_3\underline{f}_2 + h^2\Psi_{1,1}\,\underline{f}_3 + hS_{1,1}\,\underline{g}_3 + h^3\xi_{1,1}\,\underline{p}_2 + h^4\phi_{1,1}\,\underline{p}_3$$

$$\underline{\kappa}_3 = -\frac{315}{44}\,h^3\xi_{1,1}\,\underline{f}_3 - \frac{525}{44}\,h^4\phi_{1,1}\,\underline{p}_2$$

$$\underline{\kappa}_4 = -\,h\,\phi_{1,1}\,\underline{s}_3$$

The k^{th} component of the vectors $\underline{a}_1, \ldots, \underline{s}_3$ appearing in (A8) are given by

$$a_1^{(k)} = \frac{2}{5}\, n^{(k)}\, C_{55}^{(k)} \ , b_1^{(k)} = \frac{4}{5}(-1)^k\, C_{55}^{(k)} \ , c_1^{(k)} = \frac{2}{5}\, n_o^{(k)} n^{(k)} C_{55}^{(k)} \ , d_1^{(k)} = \frac{1}{5}(3n_o^{(k)2} - \frac{1}{4}) n^{(k)} C_{55}^{(k)}$$

$$e_1^{(k)} = \frac{3}{5}\,(5n_o^{(k)2} - \frac{1}{4}) n^{(k)}\, C_{55}^{(k)} \ , f_1^{(k)} = \frac{7}{120} n^{(k)3}\, C_{55}^{(k)} \ , g_1^{(k)} = \frac{7}{60}(-1)^k n^{(k)2} C_{55}^{(k)} \ , p_1^{(k)} = \frac{7}{40} n_o^{(k)} n^{(k)3}\, C_{55}^{(k)}$$

$$a_2^{(k)} = \frac{2}{5} n^{(k)} C_{13}^{(k)} \ , a_3^{(k)} = \frac{2}{5} n^{(k)} C_{33}^{(k)} \ , b_3^{(k)} = \frac{4}{5}(-1)^k\, C_{33}^{(k)} \ , c_2^{(k)} = \frac{6}{5} n_o^{(k)} n^{(k)} C_{33}^{(k)} \ , c_3^{(k)} = \frac{2}{5} n_o^{(k)} n^{(k)} C_{13}^{(k)}$$

$$d_3^{(k)} = \frac{1}{5}(3n_o^{(k)2} - \frac{1}{4}) n^{(k)}\, C_{13}^{(k)} \ , e_3^{(k)} = \frac{1}{5}(5n_o^{(k)3} - \frac{3}{4} n_o^{(k)}) n^{(k)}\, C_{13}^{(k)} \ , f_2^{(k)} = \frac{11}{350}\, n^{(k)3}\, C_{33}^{(k)} \quad \text{(A9)}$$

$$f_3^{(k)} = \frac{11}{1050} n^{(k)3}\, C_{13}^{(k)} \ , g_3^{(k)} = \frac{11}{525}(-1)^k n^{(k)2} C_{13}^{(k)} \ , p_2^{(k)} = \frac{11}{350}\, n_o^{(k)}\, n^{(k)3}\, C_{13}^{(k)}$$

$$p_3^{(k)} = \frac{11}{700}\,(5\, n_o^{(k)2} - \frac{1}{4}) n^{(k)3}\, C_{13}^{(k)} \ , s_3^{(k)} = \frac{11}{2688}\, n^{(k)5}\, C_{13}^{(k)} \ .$$

Table 1 Central Deflection $\bar{u}_3$ for Symmetric Cross-Ply Laminates in Cylindrical Bending Under Sinusoidal Loading

	$S = 4$			$S = 6$		
Number of Layers N	3	5	9	3	5	9
Exact Solution [1]	2.887	3.044	3.324	1.635	1.721	1.929
Present Theory	2.881	3.032	3.313	1.634	1.716	1.921
First-Order Zig-Zag [10]	2.907	3.018	3.231	1.636	1.702	1.875
LCW [7]	2.687	2.597	2.835	1.514	1.507	1.708

Table 2 Central Deflection $\bar{u}_3$ for Asymmetric Cross-Ply Laminates in Cylindrical Bending Under Sinusoidal Loading

	$S = 4$		$S = 6$	
Number of Layers N	4	8	4	8
Exact Solution [1]	4.181	3.724	2.562	2.224
Present Theory	4.105	3.625	2.519	2.181
First-Order Zig-Zag [10]	3.316	3.225	2.107	1.934
LCW [7]	3.587	3.189	2.242	1.979

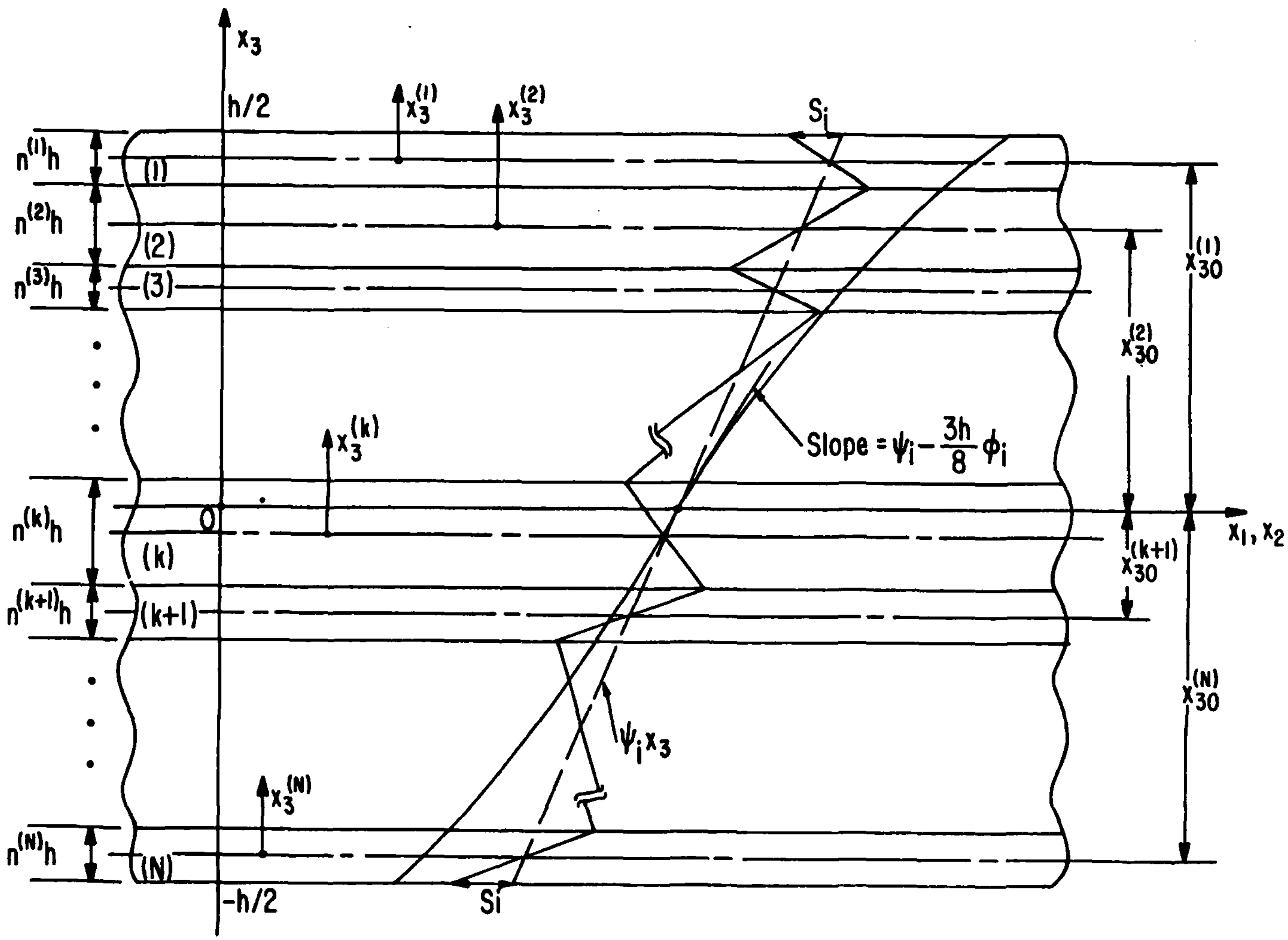

FIGURE 1: Plate geometry, coordinate system and trial in-plane displacements.

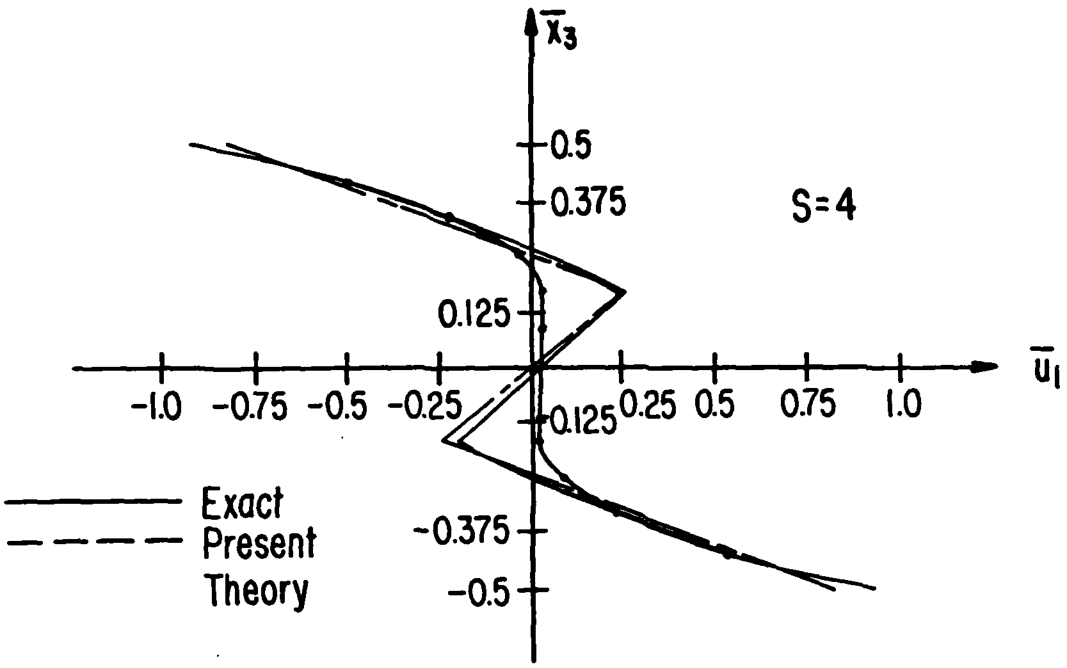

FIGURE 2a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 3-layer cross-ply laminate for S = 4.

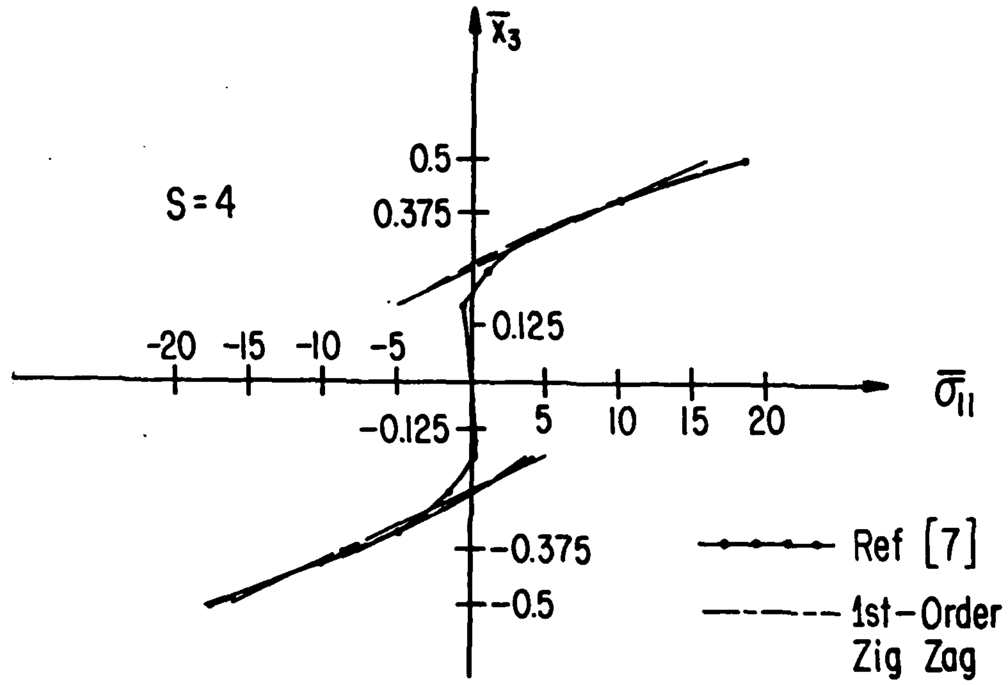

FIGURE 2b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 3-layer cross ply laminate for S = 4.

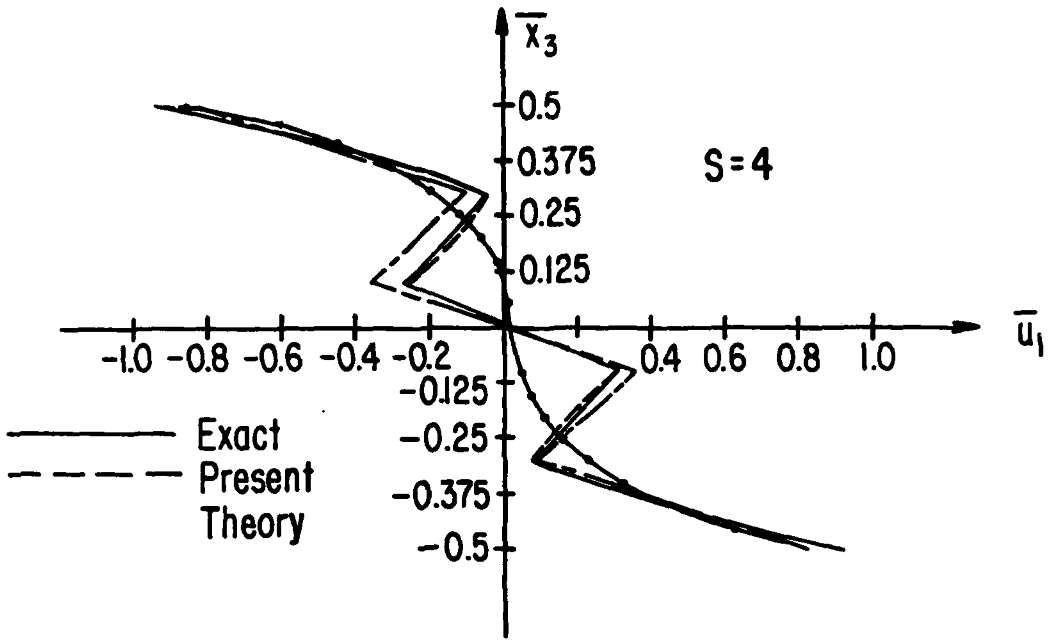

FIGURE 3a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 5-layer cross-ply laminate for S = 4.

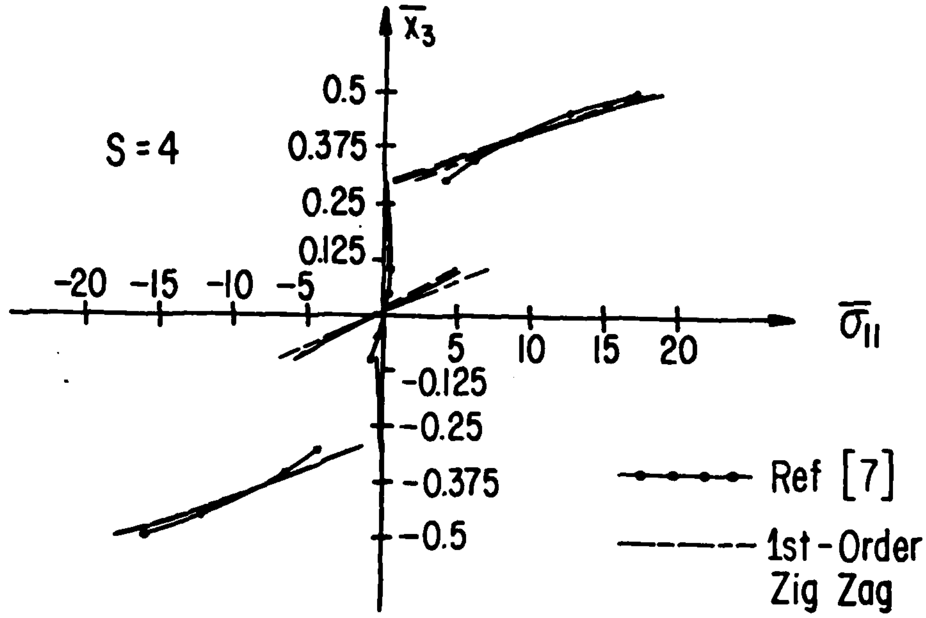

FIGURE 3b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 5-layer cross-ply laminate for S = 4.

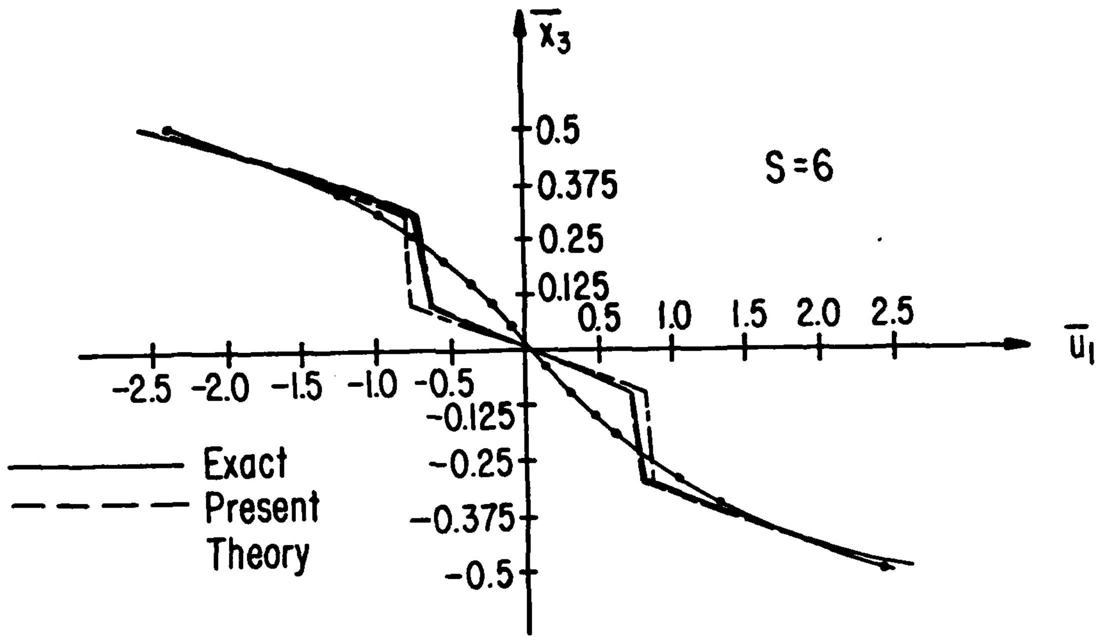

FIGURE 4a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 5-layer cross-ply laminate for S = 6.

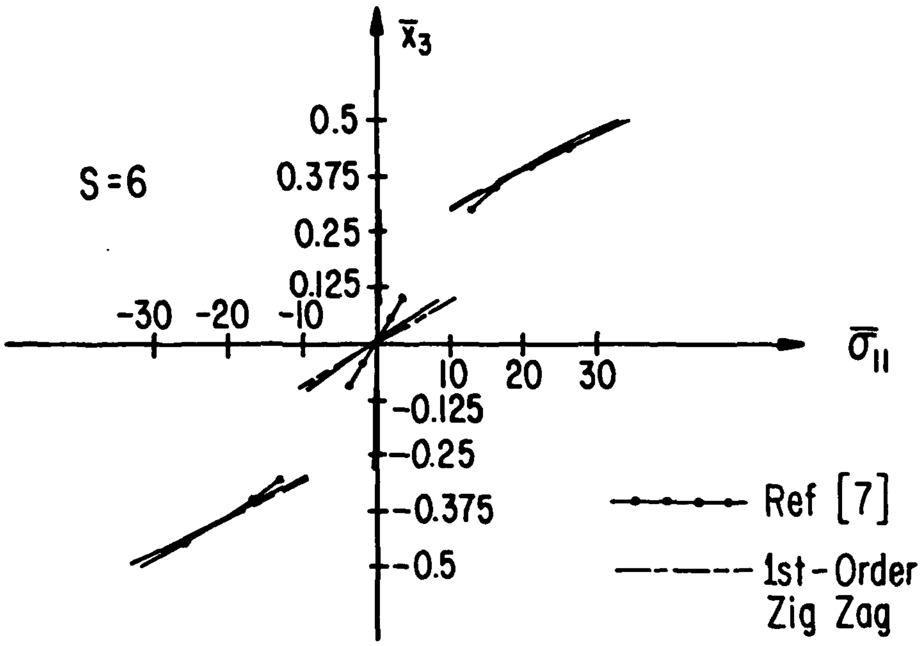

FIGURE 4b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 5-layer cross-ply laminate for S = 6.

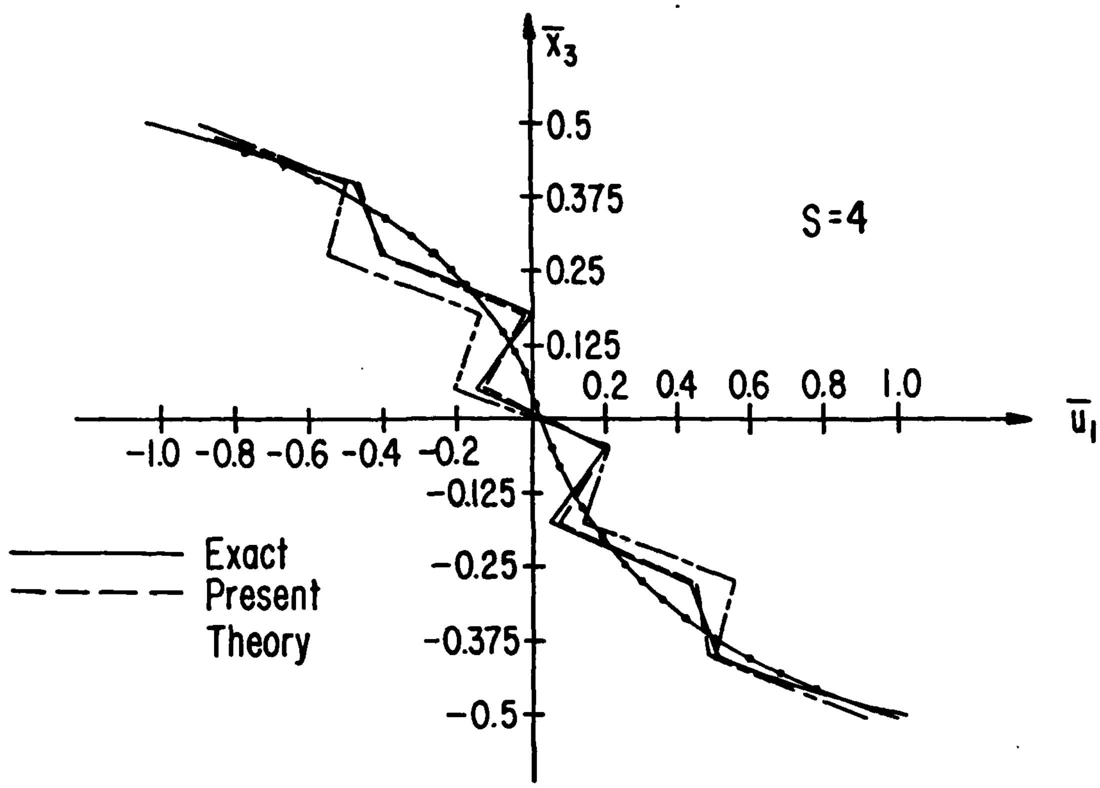

FIGURE 5a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 9-layer cross-ply laminate for S = 4.

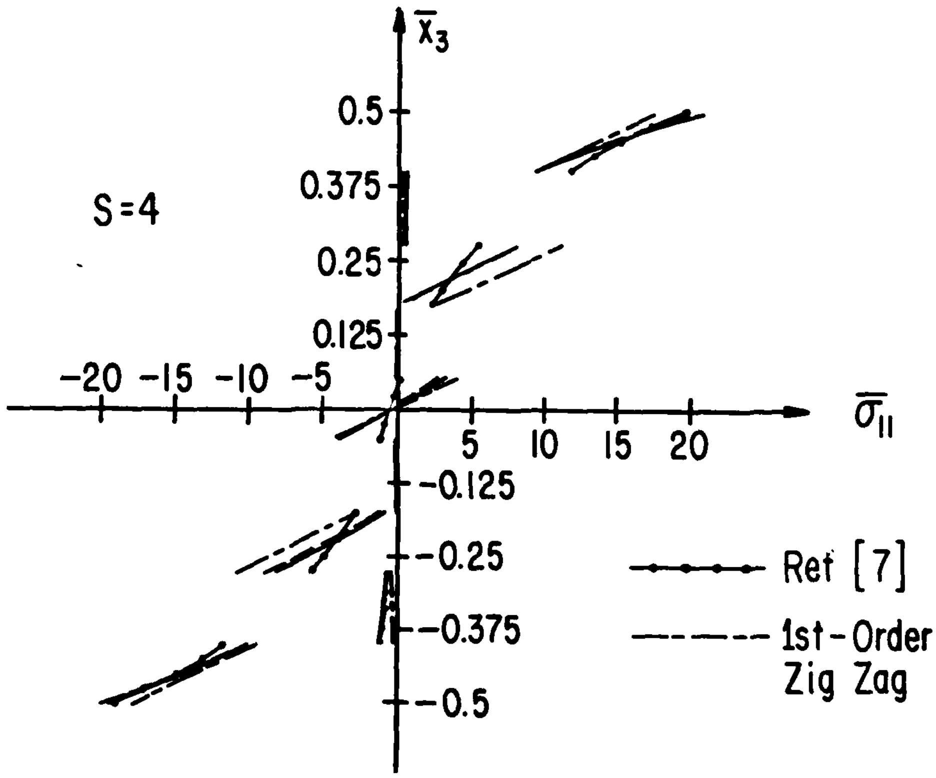

FIGURE 5b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 9-layer cross-ply laminate for S = 4.

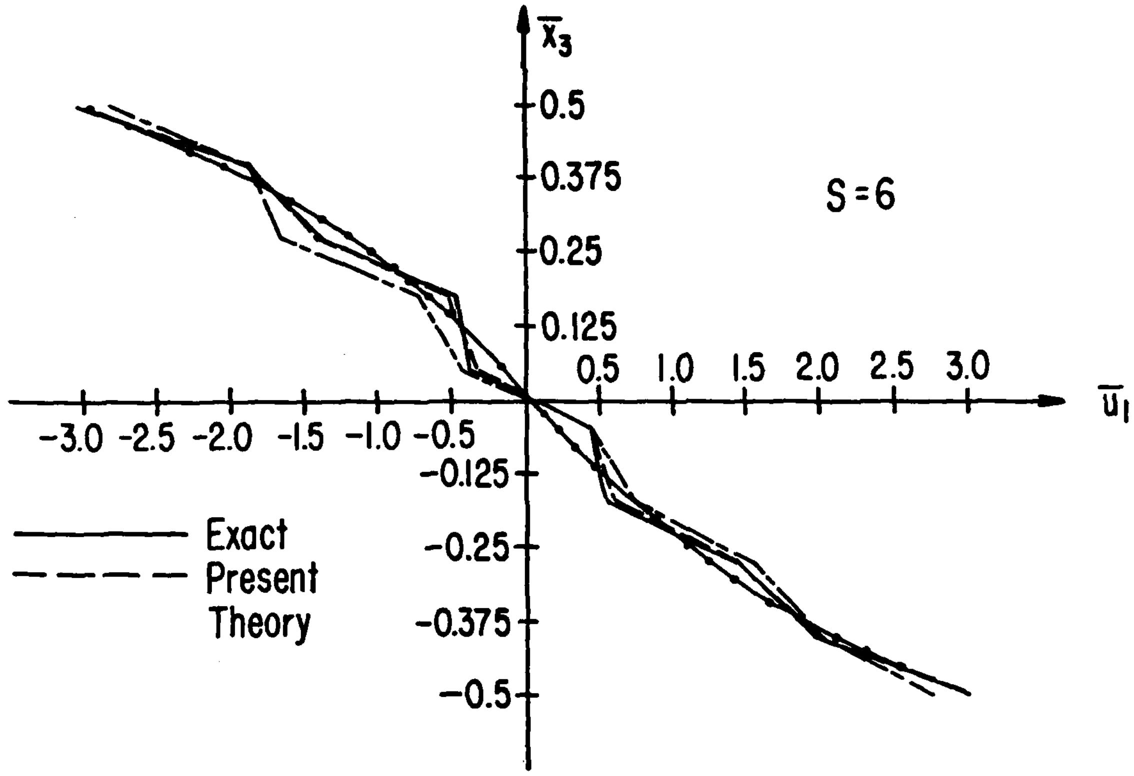

FIGURE 6a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of a symmetric 9-layer cross-ply laminate for S = 6.

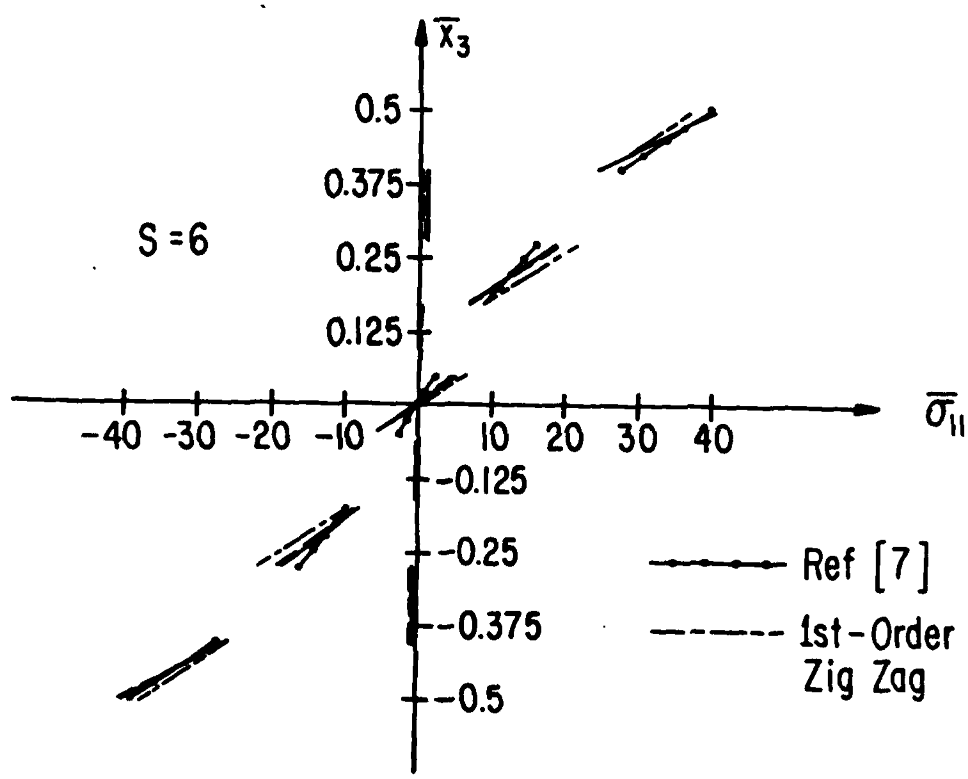

FIGURE 6b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of a symmetric 9-layer cross-ply laminate for S =6.

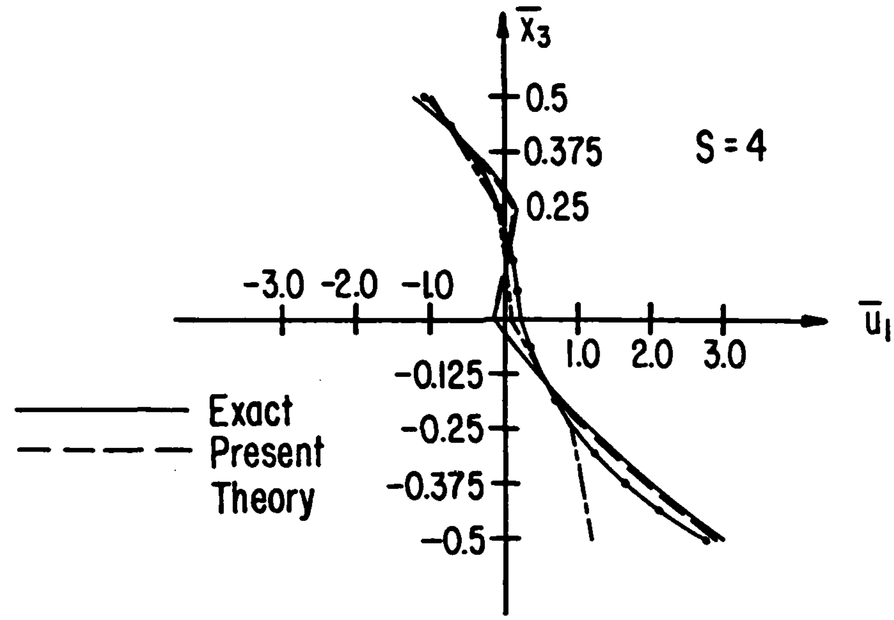

FIGURE 7a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 4-layer cross-ply laminate for S = 4.

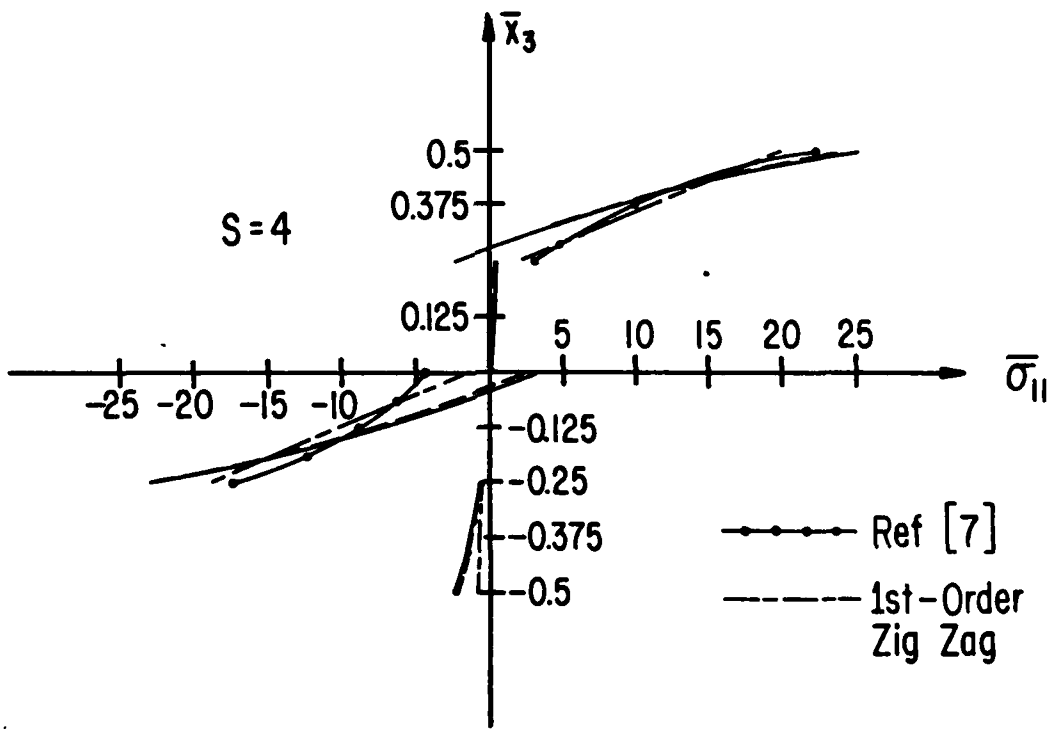

FIGURE 7b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 4-layer cross-ply laminate for S = 4.

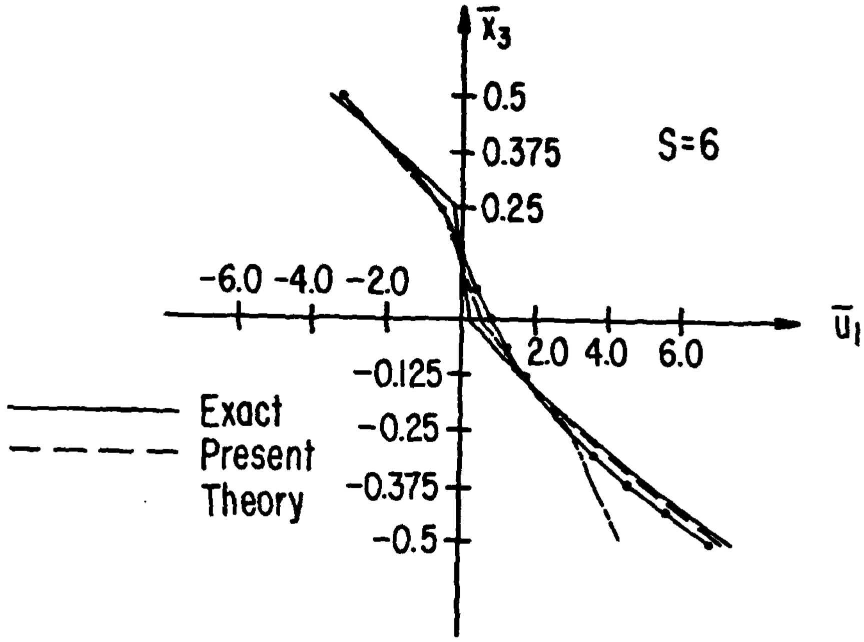

FIGURE 8a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 4-layer cross-ply laminate for S = 6.

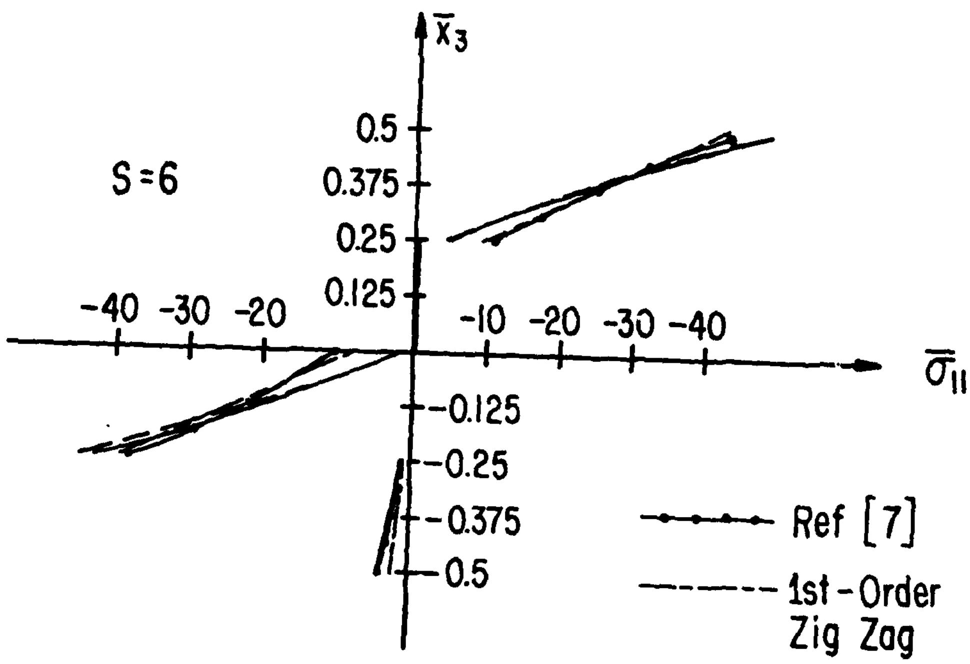

FIGURE 8b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 4-layer cross-ply laminate for S = 6.

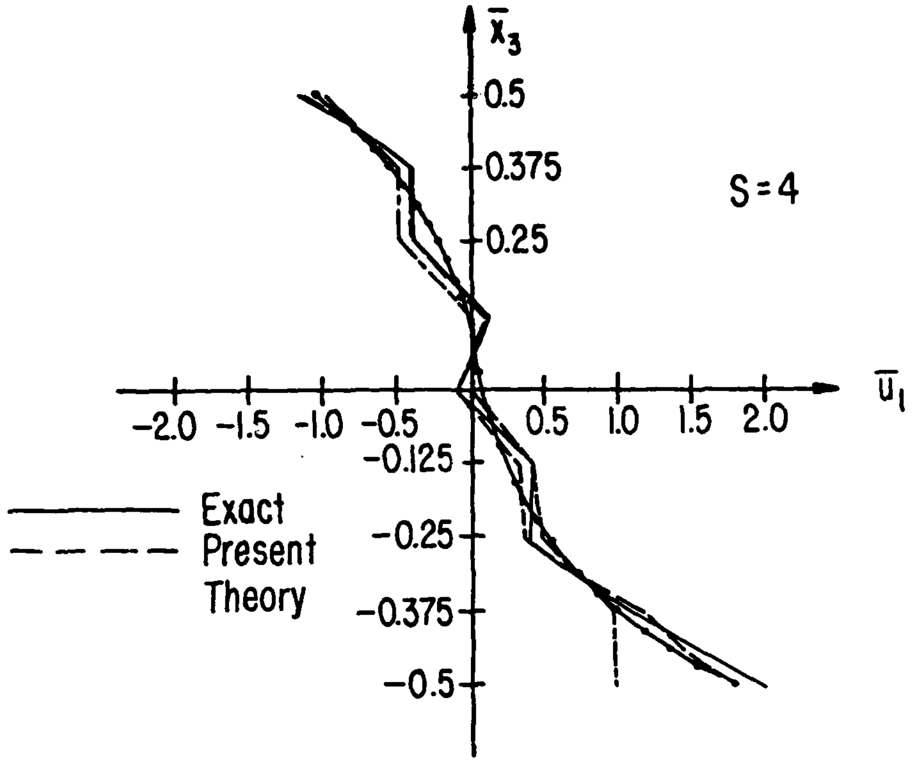

FIGURE 9a: Thickness variation of in-plane displacement $\bar{u}_1^{(k)}$ of an asymmetric 8-layer cross-ply laminate for S = 4.

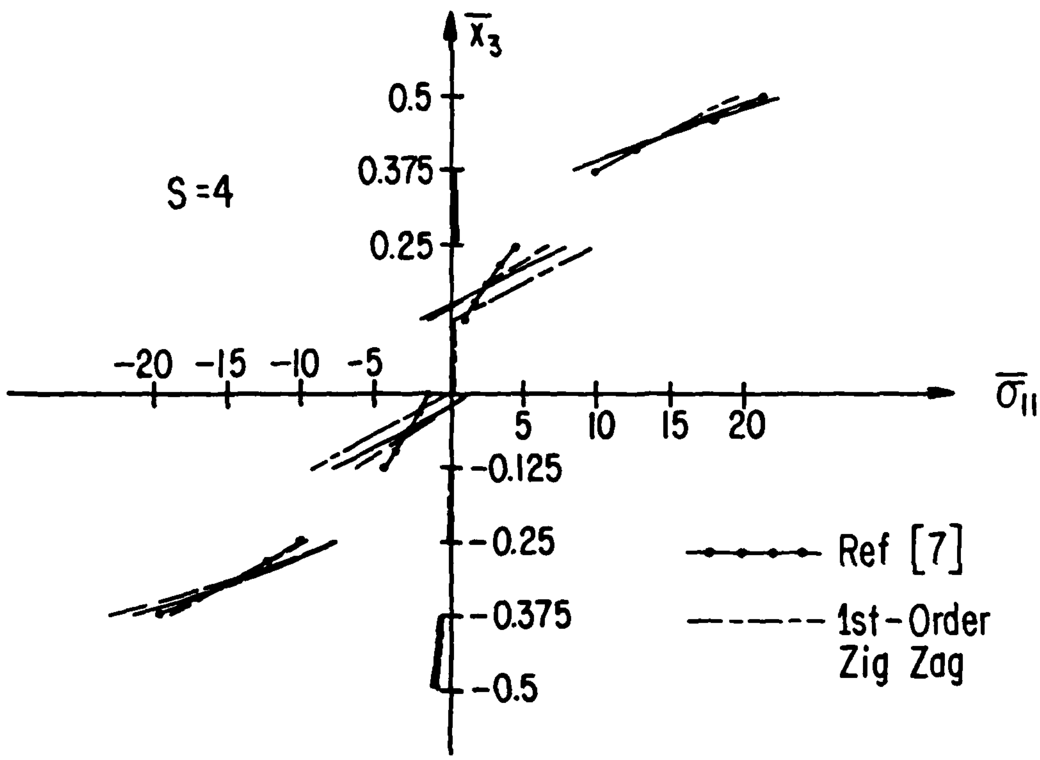

FIGURE 9b: Thickness variation of normal stress $\bar{\sigma}_{11}^{(k)}$ of an asymmetric 8-layer cross-ply laminate for S = 4.

An Improved Laminated Composite Plate Theory

by

Hidenori Murakami[1], A.M. ASCE and Alberto Toledano[2]

Abstract

Based upon Reissner's new mixed variational principle (5) for displacements and <u>some</u> stresses, a new high-order laminated plate theory was developed in order to improve the approximation of in-plane variables of the shear deformable laminated composite plate theories. To this end, a zig-zag shaped C^0 function and Legendre polynomials were introduced into the assumed in-plane displacement distributions across the plate thickness. A comparison with the exact solution obtained for symmetric 3- and 5-layer and asymmetric 4-layer cross-ply laminates indicates that the present theory provides a tool with which one may efficiently study the extraordinary skin action in laminated composite plates.

Introduction

Sandwich constructions and multilayered thick laminates exhibit under combined loads, a very different mechanical response from homogeneous isotropic plates. One such effect is the stressed skin action which may be caused by either the drastic change in material properties of the laminae or by the slip at the interface of adjacent laminae. In order to facilitate a tool with which one may study the former type of skin action laminated composite plate theories were closely examined. In a series of papers, Pagano (3,4) derived exact elasticity solutions for bidirectional composites for the problems of cylindrical bending and simply supported rectangular plates. Pagano showed the importance of incorporating transverse shear deformation effects for the accurate estimation of plate lateral deflection. Further, he showed that the linear variation of in-plane displacements across the plate thickness adopted in the classical Kirchhoff (CPT) and Reissner-Mindlin shear deformable (FSD) plate theories may not be appropriate to simulate stressed skin action in composite laminates. In addition, Whitney and Pagano (7) and Whitney (8) pointed out that the inaccuracies of CPT at low span-to-thickness ratios for determining in-plane stresses are not alleviated by the introduction of shear deformations.

The purpose of the present paper is to point out the differences in bending responses of laminated composite plates when several displacement approximations are introduced, with emphasis on the stressed skin action. A new high-order laminated plate theory, with a first-order theory as limiting case is derived using Reissner's (5) new mixed variational principle. It is a variational principle for arbitrary displacements and transverse stresses only, in which the original 3-dimensional stress-strain relations can be used. The unique features of the present theory are the inclusion in the assumed in-plane displacement variations across the plate thickness of: 1) a zig-zag shaped C^0-function; and, 2) Legendre polynomials. A comparison of the present theory with Pagano's (3) exact elasticity solution for symmetric 3- and 5-layer and asymmetric 4-layer cross-ply laminates, indicates that the in-plane responses are more accurately estimated by the inclusion of the zig-zag shaped function and the Legendre polynomials than by using smooth, C^1-interpolation functions (CPT, FSD,1).

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Problem Statement

Consider an N-layer laminated composite plate of uniform thickness h, as shown in

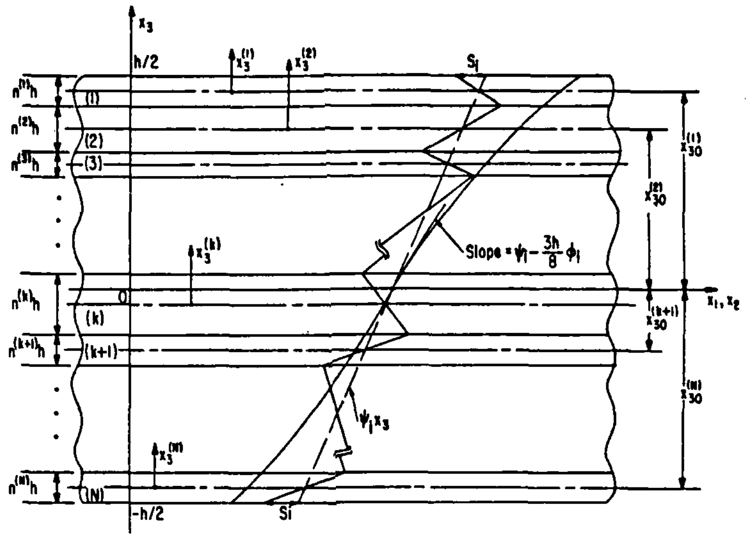

Fig. 1. Plate Geomeetry and In-Plane Trial Displacement Field

Fig. 1. A Cartesian coordinate system is selected such that the middle surface of the plate occupies a domain D in the x_1, x_2-plane, the x_3-axis being perpendicular to this plane. The notation $(\ )^{(k)}$, $k = 1,2, \cdots ,N$ is used to designate quantities associated with the k th-layer. The thickness of each layer is $n^{(k)}h$, such that the volume fractions $n^{(k)}$ satisfy the relation

$$\sum_{k=1}^{N} n^{(k)} = 1 \ . \tag{1}$$

Unless otherwise specified, the usual Cartesian indicial notation is employed where latin and greek indices range from 1 to 3 and 1 to 2, respectively. Repeated indices imply the summation convention and $(\ )_{,i}$ is used to denote partial differentiation with respect to x_i.

With the help of the foregoing notation, the governing equations for the displacement vector $u_i^{(k)}$ and stress tensor $\sigma_{ij}^{(k)}$ associated with the k th-layer are:

a) Equilibrium Equations

$$\sigma_{ji,j}^{(k)} + f_i^{(k)} = 0 \quad ; \ \sigma_{ij}^{(k)} = \sigma_{ji}^{(k)} \tag{2}$$

where f_i are the body forces;

b) Constitutive Equations For Orthotropic Layers

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}^{(k)} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 \\ 0 & 0 & \tilde{C}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} C_{13}/C_{33} \\ C_{23}/C_{33} \\ 0 \end{bmatrix}^{(k)} \sigma_{33}^{(k)} \tag{3a}$$

$$\begin{bmatrix} e_{33} \\ 2e_{23} \\ 2e_{31} \end{bmatrix}^{(k)} = - \begin{bmatrix} C_{13}/C_{33} & C_{23}/C_{33} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{(k)} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}^{(k)} + \begin{bmatrix} 1/C_{33} & 0 & 0 \\ 0 & 1/C_{44} & 0 \\ 0 & 0 & 1/C_{55} \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}^{(k)} \tag{3b}$$

where C_{ij} are the elastic constants and $\tilde{C}_{ij}$ $(i,j = 1,2,6)$ represent the reduced stiffnesses introduced by Whitney and Pagano (7);

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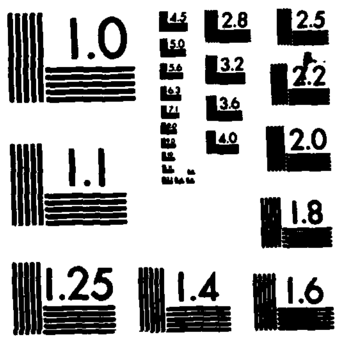

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c) Strain-Displacement Relations

$$e_{ij}^{(k)} = \frac{1}{2}\left(u_{i,j}^{(k)} + u_{j,i}^{(k)}\right) \;; \tag{4}$$

d) Interface Continuity Conditions

$$u_i^{(k)} = u_i^{(k+1)} \;,\; \sigma_{3i}^{(k)} = \sigma_{3i}^{(k+1)} \;;\; k = 1,2,..,N-1 \;; \tag{5}$$

e) Upper and Lower Surface Stress Conditions

$$\sigma_{3i}^{(1)} = T_i^+ \quad \text{on } x_3 = \frac{h}{2} \tag{6a}$$

$$\sigma_{3i}^{(N)} = T_i^- \quad \text{on } x_3 = -\frac{h}{2} \;. \tag{6b}$$

The purpose of the following analysis is to develop a laminated plate theory that will improve the assumed variation of in-plane displacements through the plate thickness and that will also account for transverse shear strains. To this end, Reissner's mixed variational principle (5) for displacements and <u>some</u> stresses was applied to the N-layer composite plate:

$$\iint_D \left[\sum_k \int_{A^{(k)}} \left\{ \delta e_{ij}^{(k)} \sigma_{ij}^{(k)} + [u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - 2e_{3\alpha}^{(k)}(\cdots)]\delta\tau_{3\alpha}^{(k)} + [u_{3,3}^{(k)} - e_{33}^{(k)}(\cdots)]\delta\tau_{33}^{(k)} \right\} dx_3 \right] dx_1\, dx_2$$

$$= \iint_D \left[\sum_k \int_{A^{(k)}} \delta u_i^{(k)} f_i^{(k)} \, dx_3 \right] dx_1\, dx_2 + \int_{\partial D_T} \left[\sum_k \int_{A^{(k)}} \delta u_i^{(k)} \overset{\nu}{T}_i \, dx_3 \right] ds \tag{7}$$

$$+ \iint_D \left[\delta u_i^{(1)}(x_1,x_2,\tfrac{h}{2}) T_i^+ - \delta u_i^{(N)}(x_1,x_2 - \tfrac{h}{2})\, T_i^- \right] dx_1\, dx_2$$

where ∂D_T denotes the boundary of D with outward normal ν_α on which tractions $\overset{\nu}{T}_i$ are specified and $A^{(k)}$ represents the x_3-domain occupied by the k th-layer. Also $e_{3i}(...)$ implies the appropriate right-hand side of Eq. 3b. Due to the nature of Reissner's mixed variational principle, Eqs. 3a are taken to be the definitions of $\sigma_{\alpha\beta}^{(k)}$ used in connection with Eq. 7.

Trial Displacement Field, Transverse and Normal Stresses

The present laminated plate theory which accounts for transverse shear strains, is a high-order theory obtained by superposing a zig-zag in-plane displacement variation of amplitude $S_i(x_1,x_2)$ across the plate thickness to the Legendre polynomials of order $n = 1, 2, 3$ in the variable x_3 (see Fig. 1). A first-order theory (2) can be recovered from the following equations by disregarding the underlined terms and numbered equations.

The suitable trial functions to be used in Reissner's mixed variational principle Eq. 7 are chosen as:

a) Trial Displacement Field:

$$\begin{bmatrix} u_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) \\ u_3(x_1,x_2,x_3) \end{bmatrix}^{(k)} = \begin{bmatrix} U_1 & \Psi_1 & S_1 & \underline{\xi_1} & \underline{\phi_1} \\ U_2 & \Psi_2 & S_2 & \underline{\xi_2} & \underline{\phi_2} \\ U_3 & \underline{\Psi_3} & \underline{S_3} & \underline{\xi_3} & \underline{0} \end{bmatrix} \begin{bmatrix} 1 \\ (\frac{h}{2})P_1(\zeta) \\ (-1)^k \frac{2}{n^{(k)}h} x_3^{(k)} \\ (\frac{h}{2})^2 P_2(\zeta) \\ (\frac{h}{2})^3 P_3(\zeta) \end{bmatrix} \tag{8}$$

where $\zeta \equiv \dfrac{2x_3}{h}$ and $P_n(\zeta)$ are the Legendre Polynomials of order n. It is also understood that U_i, Ψ_i, S_i, ξ_i and ϕ_α are functions of x_1 and x_2 only. $x_3^{(k)}$ is a local x_3-coordinate system with its origin at the center $x_{30}^{(k)}$ of the k th-layer, i.e.

$$x_3^{(k)} \equiv x_3 - x_{30}^{(k)} \quad ; \tag{9}$$

b) Trial Transverse and Normal Stresses:

$$\begin{aligned}\tau_{3i}^{(k)}(x_1,x_2,x_3) = {} & Q_i^{(k)}(x_1,x_2)F_1(z) + R_i^{(k)}(x_1,x_2)[(\delta_{i1}+\delta_{i2})F_2(z)+\delta_{i3}F_6(z)] + J_i^{(k)}(x_1,x_2)F_3(z) \\ & + I_3^{(k)}(x_1,x_2)\delta_{i3}F_7(z) + [T_i^{(k-1)}(x_1,x_2) + T_i^{(k)}(x_1,x_2)]F_4(z) \\ & + [T_i^{(k-1)}(x_1,x_2) - T_i^{(k)}(x_1,x_2)][(\delta_{i1}+\delta_{i2})F_5(z) + \delta_{i3}F_8(z)]\end{aligned} \tag{10}$$

where δ_{ij} is the Kronecker delta and

$$\begin{bmatrix} n^{(k)}h\ F_1(z) \\ (n^{(k)}h)^2\ F_2(z)/30 \\ (n^{(k)}h)^3\ F_2(z)/105 \\ F_4(z) \\ F_5(z) \\ (n^{(k)}h)^2\ F_6(z)/105 \\ (n^{(k)}h)^4\ F_7(z)/315 \\ F_8(z) \end{bmatrix} = \begin{bmatrix} 0 & 105 & 0 & -75/2 & 0 & 45/2 \\ 0 & 0 & 0 & -6 & 0 & 3/2 \\ 0 & 0 & -4 & 0 & 1 & 0 \\ 0 & -20 & 0 & 6 & 0 & -1/4 \\ 0 & 35 & 0 & -15/2 & 0 & 3/16 \\ 0 & 0 & 0 & 3 & 0 & -1/4 \\ 0 & 0 & 10 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 36 & 0 & -14 & 0 & 5/4 & 0 \\ -112 & 0 & 40 & 0 & -3 & 0 \\ 126 & 0 & -35 & 0 & 15/8 & 0 \end{bmatrix} \begin{bmatrix} z^5 \\ z^4 \\ z^3 \\ z^2 \\ z \\ 1 \end{bmatrix} \tag{11}$$

$$z \equiv \frac{x_3^{(k)}}{n^{(k)}h} \quad , \quad -\frac{1}{2} \leqslant z \leqslant \frac{1}{2} \ .$$

Also ,
$$(Q_i^{(k)}, R_i^{(k)}, J_i^{(k)}) \equiv \int_{A^{(k)}} (1, x_3^{(k)}, x_3^{(k)2})\, \tau_{3i}^{(k)}\ dx_3 \tag{12a}$$

$$I_3^{(k)} \equiv \int_{A^{(k)}} x_3^{(k)3}\, \tau_{33}^{(k)}\ dx_3 \ . \tag{12b}$$

In Eq. 10 $T_i^{(k-1)}$ and $T_i^{(k)}$ are the values of τ_{3i} at the top and bottom surfaces of the k th-layer respectively. From Eqs. 6

$$T_i^{(0)} = T_i^+ \ \text{ and } \ T_i^{(N)} = T_i^- \ . \tag{13}$$

In Eq. 11, for the functions $F_i(z)$, $i = 1, 4, 5$ two rows of coefficients appear: the upper one corresponds to the high-order theory while the lower one corresponds to the first-order theory which, in addition to disregarding the underlined terms in Eq. 8, is obtained by setting $\tau_{33}^{(k)} \equiv 0$.

For each of the theories considered, the degree of the polynomials $F_i(z)$, $i = 1, \ldots, 8$ appearing in Eq. 11 is consistent with the order of approximation used for the displacements $u_i^{(k)}$ in Eq. 8.

Laminated Plate Equations

Substituting Eqs. 8 and 10 into Eq. 7, using Gauss' Theorem and the orthogonality property of the Legendre polynomials one obtains:

a) Equilibrium Equations

$$N_{\alpha i,\alpha} + T_i^+ - T_i^- + F_i^N = 0 \tag{14a}$$

$$M_{\alpha i,\alpha} - N_{3i} + \frac{h}{2}\,(T_i^+ + T_i^-) + F_i^M = 0 \tag{14b}$$

$$Z_{\alpha i,\alpha} - K_{3i} - (T_i^+ - (-1)^N\, T_i^-) + F_i^Z = 0 \tag{14c}$$

$$L_{\alpha i,\alpha} - 3M_{3i} + \frac{h^2}{4}\,(T_i^+ - T_i^-) + F_i^L = 0 \tag{14d}$$

$$P_{\beta\alpha,\beta} - (5L_{3\alpha} + \frac{h^2}{4}\,N_{3\alpha}) + \frac{h^3}{8}\,(T_\alpha^+ + T_\alpha^-) + F_\alpha^P = 0 \tag{14e}$$

where

$$\begin{bmatrix} N_{\alpha\beta}, & M_{\alpha\beta}, & Z_{\alpha\beta}, & L_{\alpha\beta}, & P_{\alpha\beta} \\ F_i^N, & F_i^M, & F_i^Z, & F_i^L, & F_i^P \end{bmatrix} \equiv \sum_{k=1}^{N} \int_{A^{(k)}}$$

$$[1,\ \frac{h}{2}\,P_1(\zeta), (-1)^k\,\frac{2x_3^{(k)}}{n^{(k)}h}\ ,\ (\frac{h}{2})^2\,P_2(\zeta)\ ,\ (\frac{h}{2})^3 P_3(\zeta)] \begin{bmatrix} \sigma_{\alpha\beta}^{(k)} \\ f_i^{(k)} \end{bmatrix} dx_3 \tag{15a,b}$$

$$(N_{3i}\ , M_{3i}\ , K_{3i}\ , Z_{3i}\ , L_{3i}) \equiv \sum_{k=1}^{N} \int_{A^{(k)}} \Bigg[1,\ \frac{h}{2}\,P_1(\zeta), (-1)^k\,\frac{2}{n^{(k)}h}\ ,$$

$$(-1)^k\,\frac{2}{n^{(k)}h}\,x_3^{(k)}\ ,\ (\frac{h}{2})^2\,P_2(\zeta)\Bigg]\,\tau_{3i}^{(k)}\,dx_3\ ; \tag{15c}$$

b) Constitutive Equations

- For Transverse Stresses

$$Q_\alpha^{(k)} - \frac{8\,J_\alpha^{(k)}}{(n^{(k)}h)^2} + a_1\,n^{(k)}h\,\Big[T_\alpha^{(k-1)} + T_\alpha^{(k)}\Big] = b_1\,h\,n^{(k)}\,\tilde{C}_\alpha^{(k)}\,\Big[U_{3,\alpha} + \Psi_\alpha + S_\alpha(-1)^k$$

$$\frac{2}{n^{(k)}h} + h\,n_o^{(k)}(\Psi_{3,\alpha} + 3\xi_\alpha) + \frac{h^2}{2}\,(3\,n_o^{(k)2} - \frac{1}{4})\,\xi_{3,\alpha} + \frac{3h^2}{2}\,(5\,n_o^{(k)2} - \frac{1}{4})\phi_\alpha\Big] \tag{16a}$$

$$\frac{1}{h}\,R_\alpha^{(k)} - \frac{n^{(k)2}h}{40}\,(T_\alpha^{(k-1)} - T_\alpha^{(k)}) = \frac{7h^2}{120}\,n^{(k)3}\,\tilde{C}_\alpha^{(k)}\,\Big[\Psi_{3,\alpha} + 3\xi_\alpha + S_{3,\alpha}\,(-1)^k\,\frac{2}{n^{(k)}h}$$

$$+ 3h\,n_o^{(k)}\,(\xi_{3,\alpha} + 5\phi_\alpha)\Big] \tag{16b}$$

$$Q_\alpha^{(k)} - \frac{14J_\alpha^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{12}\,(T_\alpha^{(k-1)} + T_\alpha^{(k)}) = -\frac{3h^2}{40}\,n^{(k)3}\,\tilde{C}_\alpha^{(k)}\,(\xi_{3,\alpha} + 5\phi_\alpha) \tag{16c}$$

$$\frac{1}{\tilde{C}_\alpha^{(k)}}\left[a_2\,Q_\alpha^{(k)} + \frac{5\,J_\alpha^{(k)}}{3(n^{(k)}h)^2} - \frac{3\,R_\alpha^{(k)}}{7\,n^{(k)}h}\right] + \frac{1}{\tilde{C}_\alpha^{(k+1)}}\Bigg[a_2\,Q_\alpha^{(k+1)}$$

$$+ \frac{5\,J_\alpha^{(k+1)}}{3\,(n^{(k+1)}h)^2} + \frac{3\,R_\alpha^{(k+1)}}{7\,n^{(k+1)}h}\Bigg] \tag{16d}$$

$$= h\left[\frac{b_2\,n^{(k)}}{\tilde{C}_\alpha^{(k)}}\,T_\alpha^{(k-1)} + b_3\left(\frac{n^{(k)}}{\tilde{C}_\alpha^{(k)}} + \frac{n^{(k+1)}}{\tilde{C}_\alpha^{(k+1)}}\right)T_\alpha^{(k)} + b_2\,\frac{n^{(k+1)}}{\tilde{C}_\alpha^{(k+1)}}\,T_\alpha^{(k+1)}\right];$$

- For Normal Stresses (High-Order Theory Only)

$$Q_3^{(k)} - \frac{8J_3^{(k)}}{(n^{(k)}h)^2} + \frac{n^{(k)}h}{30}\,(T_3^{(k-1)} + T_3^{(k)}) = \frac{2h}{5}\,n^{(k)}\,C_{33}^{(k)}\,\Big[\Psi_3 + S_3(-1)^k\,\frac{2}{n^{(k)}h} + 3h\,n_o^{(k)}\,\xi_3\Big]$$

$$+\frac{2h}{5}\,n^{(k)}\left[\bar{U}+h\,n_o^{(k)}\,\bar{\Psi}+\frac{h^2}{2}\,(3\,n_o^{(k)2}-\frac{1}{4})\,\bar{\xi}+\frac{h^3}{2}\left[5\,n_o^{(k)3}-\frac{3}{4}\,n_o^{(k)}\right]\bar{\phi}\right] \tag{17a}$$

$$\frac{1}{h}\,R_3^{(k)}-\frac{32\,I_3^{(k)}}{5\,n^{(k)2}h^3}+\frac{n^{(k)2}h}{140}\,(T_3^{(k-1)}-T_3^{(k)})=\frac{11}{350}\,h^2\,n^{(k)3}\,C_{33}^{(k)}\,\xi_3$$

$$+\frac{11}{1050}\,h^2 n^{(k)3}\left[\bar{\Psi}+(-1)^k\,\frac{2}{n^{(k)}h}\,\bar{S}+3h\,n_o^{(k)}\,\bar{\xi}+\frac{3h^2}{2}\,(5n_o^{(k)2}-\frac{1}{4})\bar{\phi}\right] \tag{17b}$$

$$Q_3^{(k)}-\frac{14\,J_3^{(k)}}{(n^{(k)}h)^2}+\frac{n^{(k)}h}{12}\,(T_3^{(k-1)}+T_3^{(k)})=-\,\frac{3h^3}{40}\,n^{(k)3}\,[\bar{\xi}+5h\,n_o^{(k)}\,\bar{\phi}\,] \tag{17c}$$

$$\frac{1}{h}\,R_3^{(k)}-\frac{15\,I_3^{(k)}}{2n^{(k)2}h^3}+\frac{n^{(k)2}h}{96}\,(T_3^{(k-1)}-T_3^{(k)})=\frac{-\,11h^4}{2688}\,n^{(k)5}\,\bar{\phi} \tag{17d}$$

$$\frac{-\,11}{12}\left[\frac{Q_3^{(k)}}{C_{33}^{(k)}}+\frac{Q_3^{(k+1)}}{C_{33}^{(k+1)}}\right]+\frac{15}{2h}\left[\frac{R_3^{(k)}}{n^{(k)}C_{33}^{(k)}}-\frac{R_3^{(k+1)}}{n^{(k+1)}C_{33}^{(k+1)}}\right]$$

$$+\frac{5}{3h^2}\left[\frac{J_3^{(k)}}{n^{(k)2}C_{33}^{(k)}}+\frac{J_3^{(k+1)}}{n^{(k+1)2}\,C_{33}^{(k+1)}}\right]$$

$$\frac{-\,70}{11h^3}\left[\frac{I_3^{(k)}}{n^{(k)3}C_{33}^{(k)}}-\frac{I_3^{(k+1)}}{n^{(k+1)3}\,C_{33}^{(k+1)}}\right]=\frac{h}{18}\left[\frac{n^{(k)}}{C_{33}^{(k)}}\,T_3^{(k-1)}+10\left(\frac{n^{(k)}}{C_{33}^{(k)}}\right.\right.$$

$$\left.\left.+\frac{n^{(k+1)}}{C_{33}^{(k+1)}}\right)T_3^{(k)}+\frac{n^{(k+1)}}{C_{33}^{(k+1)}}\,T_3^{(k+1)}\right] \tag{17e}$$

where in Eqs. 16a,b,c and 17a,b,c,d k ranges from 1 to N while in Eqs. 16d and 17e, k ranges from 1 to $(N-1)$. Also, no summation on α is implied in Eq. 16 and

$$\tilde{C}_\alpha^{(k)}\equiv\delta_{\alpha 1}\,C_{55}^{(k)}+\delta_{\alpha 2}\,C_{44}^{(k)}\;\;;\;\;n_o^{(k)}\equiv x_{30}^{(k)}/h\;\;. \tag{18}$$

Also, $\bar{U}\equiv C_{13}^{(k)}\,U_{1,1}+C_{23}^{(k)}\,U_{2,2}$, with analogous expressions for $\bar{\Psi}$, $\bar{S}$, $\bar{\xi}$ and $\bar{\phi}$. The coefficients a_1, b_1, a_2 b_2 and b_3 appearing in Eq. 16 are given in Table 1.

Table 1 Coefficients a_1, b_1, a_2, b_2 and b_3 Appearing in Eqs. 16

	a_1	b_1	a_2	b_2	b_3
High-Order Theory	$\frac{1}{30}$	$\frac{2}{5}$	$-\frac{1}{12}$	$-\frac{1}{126}$	$\frac{4}{63}$
First-Order Theory	$-\frac{1}{12}$	$\frac{5}{6}$	$\frac{1}{10}$	$-\frac{1}{30}$	$\frac{2}{15}$

c) Boundary Conditions

$$\text{specify}\quad U_i\quad\text{or}\quad N_{\alpha i}\,\nu_\alpha\;, \tag{19a}$$

$$\text{specify}\quad \Psi_i\quad\text{or}\quad M_{\alpha i}\,\nu_\alpha\;, \tag{19b}$$

$$\text{specify}\quad S_i\quad\text{or}\quad Z_{\alpha i}\,\nu_\alpha\;, \tag{19c}$$

$$\text{specify}\quad \xi_i\quad\text{or}\quad L_{\alpha i}\,\nu_\alpha\;, \tag{19d}$$

$$\text{specify}\quad \phi_\alpha\quad\text{or}\quad P_{\beta\alpha}\,\nu_\beta\;. \tag{19e}$$

It should be pointed out that for the first-order theory the subscript i in Eqs. 14b,c and 19b,c will assume the values 1 and 2 only.

The quantities N_{3i}, M_{3i}, K_{3i}, Z_{3i} and L_{3i}, obtained by inserting $Q_i^{(k)}$, $R_i^{(k)}$, $J_i^{(k)}$, $I_3^{(k)}$ and $T_i^{(k)}$ from Eqs. 16 and 17 into Eqs. 10 and 15c will automatically include the appropriate shear correction factors by virtue of Reissner's mixed variational principle (5). The remaining constitutive equations for $N_{\alpha\beta}$, $M_{\alpha\beta}$, $Z_{\alpha\beta}$, $L_{\alpha\beta}$ and $P_{\alpha\beta}$ are obtained by substituting Eqs. 3a, 4, 8 and 10b into Eq. 15a.

Cylindrical Bending of Laminated Plates

As an illustration of the present theory, cylindrical bending under sinusoidal loading of an infinitely long strip in the x_2-direction is considered. The plate is simply supported at the ends $x_1 = 0$ and l. The prescribed boundary conditions on the top and bottom surfaces of the plate are

$$T_1^+ = 0\,,\ T_3^+ = q \sin \frac{\pi x_1}{l} \qquad \text{on } x_3 = \frac{h}{2} \tag{20a}$$

$$T_1^- = T_3^- = 0 \qquad \text{on } x_3 = -\frac{h}{2}\,. \tag{20b}$$

The boundary conditions for the simply supported ends are, from Eqs. 19:

$$U_3 = \Psi_3 = S_3 = \xi_3 = 0 \qquad \text{at } x_1 = 0\,,\ l \tag{21a}$$

$$N_{11} = M_{11} = Z_{11} = L_{11} = P_{11} = 0 \qquad \text{at } x_1 = 0\,,\ l\,. \tag{21b}$$

Numerical Results

In order to assess the accuracy of the present theory, the cylindrical bending problem under sinusoidal loading of an infinitely long strip is considered. The exact elasticity solution was given by Pagano (3) where a 3-layer cross-ply laminate with the outer layers oriented at 0° was examined. The material properties are for the 0° layers

$$\frac{\tilde{C}_{11}}{E_T} = 25.062657\,,\ \frac{C_{13}}{E_T} = 0.335570\,,\ \frac{C_{33}}{E_T} = 1.071141\,,\ \frac{C_{55}}{E_T} = 0.5\ ; \tag{22a}$$

and for 90° layers

$$\frac{\tilde{C}_{11}}{E_T} = 1.002506\,,\ \frac{C_{13}}{E_T} = 0.271141\,,\ \frac{C_{33}}{E_T} = 1.071141\,,\ \frac{C_{55}}{E_T} = 0.2 \tag{22b}$$

where E_T is a reference modulus. Adopting Pagano's (3) nondimensionalization, the displacements and stresses are calculated in the form:

$$\bar{u}_1^{(k)} = \left(\frac{E_T}{q}\right) \frac{u_1^{(k)}(0,x_3)}{h}\,,\ \bar{u}_3^{(k)} = \left(\frac{E_T}{q}\right) \frac{100h^3}{l^4}\, u_3^{(k)}\left(\frac{l}{2}\,,0\right) \tag{23}$$

$$\bar{\sigma}_{11}^{(k)} = \frac{1}{q}\,\sigma_{11}^{(k)}\left(\frac{l}{2}\,,x_3\right)$$

$$\text{Also} \qquad \bar{x}_3 = \frac{x_3}{h}\,,\ S = \frac{l}{h} \tag{24}$$

The present high-order theory and the corresponding first-order theory are compared with Pagano's exact solution (3), Lo, Christensen and Wu's high-order model (LCW) (1) and Reissner-Mindlin first-order shear deformable theory (FSD). For each of these theories thickness variations of in-plane displacement $\bar{u}_1^{(k)}$ and normal stress $\bar{\sigma}_{11}^{(k)}$ of various symmetric and asymmetric laminates with span-to-thickness ratio S of 4 are shown in Figs. 2, 3 and 4.

For symmetric 3- and 5-layer cross-ply laminates, with layers of equal thickness, the results of the present high-order theory are in excellent agreement with Pagano's exact solution, as can be observed from Figs. 2 and 3. In particular, it has considerably improved upon

Lo, Christensen and Wu's high-order model at the interfaces and in the interior layers of the plate. Also the present first-order zig-zag theory gives reasonably good results even for the more difficult case of the symmetric 5-layer cross-ply laminate.

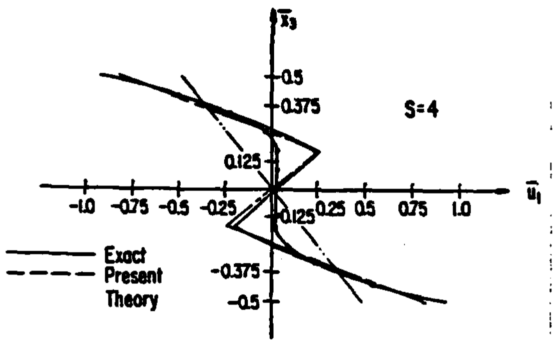

Fig. 2a. Thickness Variation of In-Plane Displacement for $N = 3$ and $S = 4$

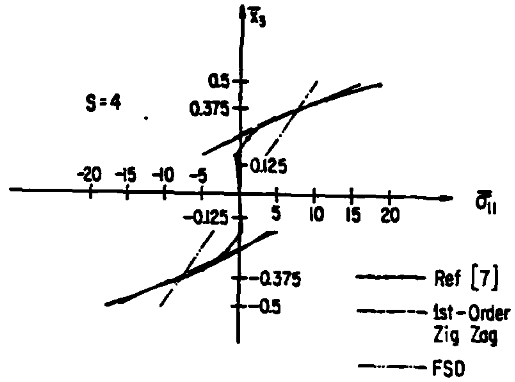

Fig. 2b. Thickness Variation of Normal Stresses for $N = 3$ and $S = 4$

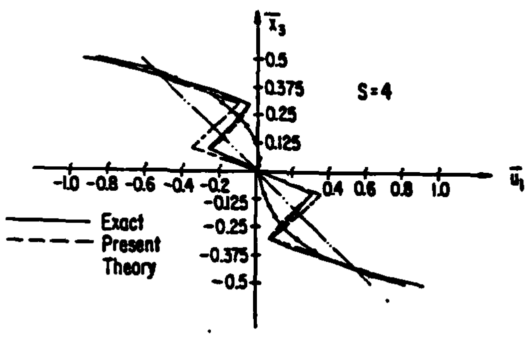

Fig. 3a. Thickness Variation of In-Plane Displacement for $N = 5$ and $S = 4$

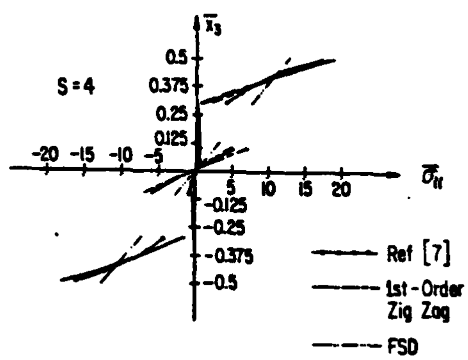

Fig. 3b. Thickness Variation of Normal Stress for $N = 5$ and $S = 4$

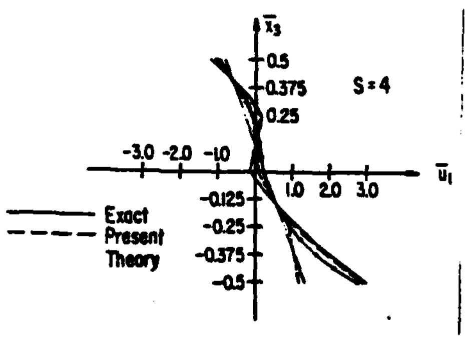

Fig. 4a. Thickness Variation of In-Plane displacement for $N = 4$ and $S = 4$

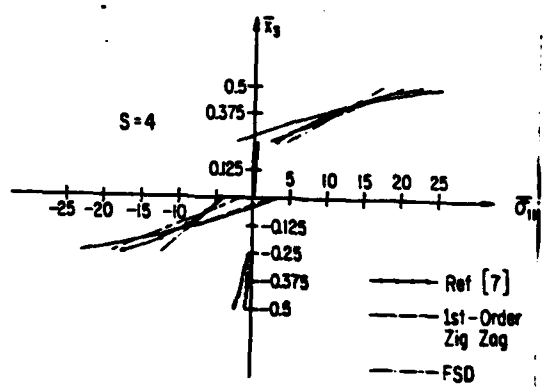

Fig. 4b. Thickness Variation of Normal Stress for $N = 4$ and $S = 4$

For an asymmetric 4-layer cross-ply laminate, with layers of equal thickness, the present first-order theory deviates considerably from the exact solution at the bottom layer of the plate, while the present high-order model still gives very good results as shown in Fig. 4. On the other hand, the discrepancies between LCW and the exact solution for both $\bar{u}_1^{(k)}$ and $\bar{\sigma}_{11}^{(k)}$ are more pronounced in the interior layers of the plate.

In all the cases considered FSD gave poor results when compared with the exact solution.

Finally, the improvement of the present theory can also be seen in the results for the central deflection $\bar{u}_3$ of the plate as shown in Table 2. It is worth noting that the present first-order, zig-zag theory gives closer values to the exact solution than LCW for symmetric laminates.

Table 2 Central Deflection $\bar{u}_3$ of Symmetric and Asymmetric Cross-Ply Laminates for $S = 4$

Number of Layers N	3	4	5
Exact Solution (3)	2.887	4.181	3.044
Present (High-Order)	2.881	4.105	3.032
Present (First-Order)	2.907	3.316	3.018
LCW (1)	2.687	3.587	2.597
FSD	2.262	3.088	2.412

Conclusion

A new high-order laminated plate theory, which accurately predicts in-plane responses of symmetric and asymmetric laminates, has been developed using Reissner's mixed variational principle (5). The improvement was accomplished by introducing into the in-plane displacement variations across the plate thickness, a zig-zag shaped C^0 function and the Legendre polynomials of order $n = 1, 2, 3$, as detailed in (6). The theory was tested by examining the problem of cylindrical bending of an infinitely long strip, whose exact solution had been given by Pagano (3). The comparison of the central deflection and in-plane displacements and normal stresses for symmetric 3- and 5-layer and asymmetric 4-layer cross-ply laminates with available first-order (FSD) and high-order (LCW) theories has shown that the present theory very accurately predicts the in-plane response even for small span-to-thickness ratios. Thus, the theory provides a tool with which one may efficiently study the extraordinary skin action in laminated composite plates.

Appendix - References

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